# A. A Binary System

A-1. Assume $a_1$ and $a_2$, are respectively, the distances of $M_1$ and $M_2$ from the center of mass:

$$\begin{cases} M_1 a_1 = M_2 a_2 \\ a_1 + a_2 = a \end{cases} \rightarrow a_1 = \frac{M_2}{M} a \ , a_2 = \frac{M_1}{M} a \ : M = M_1 + M_2$$

In the rotating coordinate system, a centrifugal potential has to be added to the gravitational potential of the two masses:

$$U = -\frac{1}{2}\omega^2 r^2 \qquad \omega = \sqrt{\frac{GM}{a^3}}$$

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{(x+a_1)^2+y^2}} - \frac{GM_2}{\sqrt{(x-a_2)^2+y^2}} - \frac{1}{2}\omega^2(x^2+y^2)$$

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{\left(x+\frac{M_2}{M}a\right)^2+y^2}} - \frac{GM_2}{\sqrt{\left(x-\frac{M_1}{M}a\right)^2+y^2}} - \frac{1}{2}\frac{GM}{a^3}(x^2+y^2)$$

A-1 (1.0 pt)

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{\left(x+\frac{M_2}{(M_1+M_2)}a\right)^2+y^2}} - \frac{GM_2}{\sqrt{\left(x-\frac{M_1}{(M_1+M_2)}a\right)^2+y^2}} - \frac{1}{2}\frac{G(M_1+M_2)}{a^3}(x^2+y^2)$$

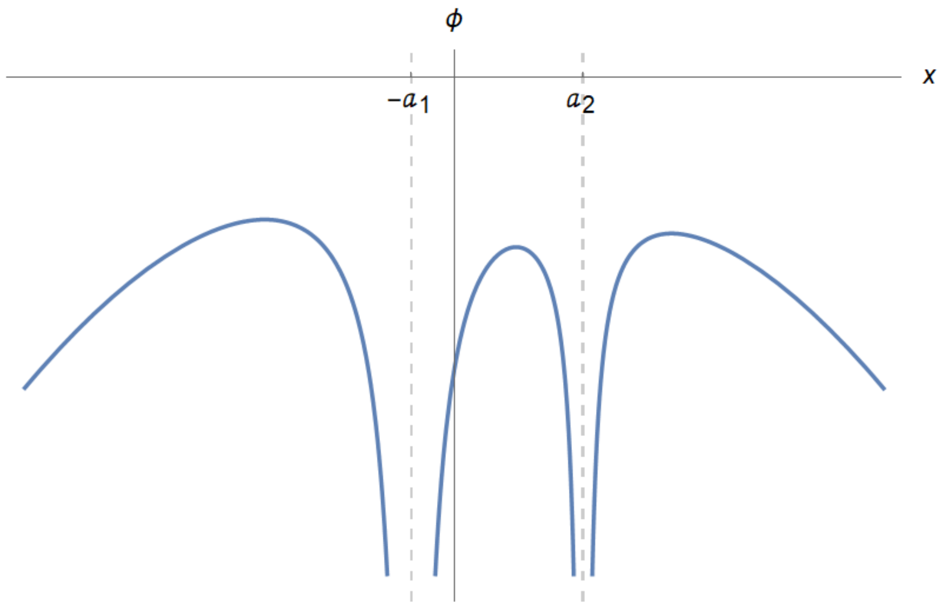
A-2. We set $y = 0$ in the previous equation, and obtain:

$$\varphi(x,0) = -\frac{GM_1}{\left|x+\frac{M_2}{M}a\right|} - \frac{GM_2}{\left|x-\frac{M_1}{M}a\right|} - \frac{1}{2}\frac{GM}{a^3}x^2$$

We draw the diagram noting that:

1. The function has asymptotes at $x = -a_1$ and $x = a_2$ , and it tends to $-\infty$ at both sides of these asymptotes.
2. The function has three maxima which are called Lagrange points.
3. The function goes to $-\infty$ for $x \rightarrow \pm\infty$

A-2 (0.7 pt)




A-3. Let $\bar{x} = x/a$, and denote the Lagrange point in the middle (between $\bar{x} = 0$ and $\bar{x} = 0.75$) by $\bar{x}_0$, we have $\frac{d\varphi}{d\bar{x}}(\bar{x}_0) = 0$. Using the given ratios:

$$\varphi(\bar{x},0) = \frac{GM}{a}\left[-\frac{\frac{3}{4}}{\left(\bar{x}+\frac{1}{4}\right)} + \frac{\frac{1}{4}}{(\bar{x}-\frac{3}{4})} - \frac{1}{2}\bar{x}^2\right]$$

Let $f(\bar{x}) = \frac{a}{GM}\frac{d\varphi}{d\bar{x}}$, then we have to solve for $f(\bar{x}_0) = 0$. We have $f(0) > 0$ and $f(0.5) < 0$, so the answer lies between 0 and 0.5. For the midpoint, we have $f(\bar{x}_0 = 0.25) > 0$ so $0.25 < \bar{x}_0 < 0.5$, so by trial and error:

$$\begin{cases} f(0) > 0 \\ f(0.5) < 0 \end{cases} \to f(0.25) > 0 \to 0.25 < \bar{x}_0 < 0.5 \to f(0.375) < 0 \to \cdots \to 0.358 < \bar{x}_0 < 0.361$$

$$\to f(0.360) > 0 \to 0.360 < \bar{x}_0 < 0.361 \to \frac{x_0}{a} = \bar{x}_0 \approx 0.36$$

So, up to two significant figures the answer is 0.36.

A-3 (0.5 pt)

$\frac{x_0}{a} = 0.36$

A-4. The angular momentum of the system is:

$$J = \mu a V = \mu a^2 \omega = \frac{M_1 M_2}{M} a^2 \sqrt{\frac{GM}{a^3}} = \sqrt{\frac{G M_1^2 M_2^2}{M}}\, a,$$

where $\mu$ is the reduced mass and $V$ is the relative velocity of the two point masses. Taking the logarithm of both sides we'll have:

$$\ln J = \frac{1}{2}\left[\ln\frac{G}{M} + 2\ln M_1 + 2\ln M_2 + \ln a\right]$$

For slowly-varying quantities we'll obtain:

$$\frac{\dot{J}}{J} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} + \frac{1}{2}\frac{\dot{a}}{a}$$

because the total mass is a constant and $\dot{M}_1 + \dot{M}_2 = 0$; therefore:

$$\frac{\dot{a}}{a} = -2\frac{\dot{M}_1}{M_1}\left(1 - \frac{M_1}{M_2}\right) \quad \rightarrow \quad \dot{a} = -2\beta a\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$$

For the period we'll have:

$$P = 2\pi\sqrt{\frac{a^3}{GM}} \;\rightarrow\; \frac{\dot{P}}{P} = \frac{3}{2}\frac{\dot{a}}{a} = -3\frac{\dot{M}_1}{M_1}\left(1 - \frac{M_1}{M_2}\right) \;\rightarrow\; \dot{P} = -6\pi\sqrt{\frac{a^3}{GM}}\beta\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$$

A-4 (0.6 pt)

$\dot{a} = -2\beta a\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$

$\dot{P} = -6\pi\sqrt{\frac{a^3}{GM}}\beta\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$

A-5. In an infinitesimally thin ring with an inner radius of $r$ and an outer radius $r + dr$, energy is leaving at a rate of $-\frac{GM_1\beta}{2r}$ and entering at a rate $-\frac{GM_1\beta}{2r} + \frac{GM_1\beta}{2r^2}dr$ . For the ring to stay in equilibrium, the excess energy of $\frac{GM_1\beta}{2r}dr$ per unit time must leave the system as radiation, so:

$$dP = \frac{GM_1\beta}{2r^2}dr = \sigma T^4 2(2\pi r dr) = 4\pi\sigma T^4 dr \;\rightarrow\; T = \left(\frac{GM_1\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}}$$

A-5 (1.0 pt)

$T = \left(\frac{GM_1\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}}$

A-6. From $P = 2\pi\sqrt{\frac{a^3}{GM}}$ we'll have:

$$a = \left[\frac{P^2 G(M_S + M_{NS})}{4\pi^2}\right]^{\frac{1}{3}}$$

Using the result of Part A.5, the temperature is:

$$T = \left(\frac{GM_{NS}\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}} = \left(\frac{500\pi\, M_{NS}\,\beta}{\sigma P^2 (M_S + M_{NS})}\right)^{\frac{1}{4}} = 9 \times 10^3 K$$

| A-6 (0.5 pt) |
|---|
| $T = 9 \times 10^3\ K$ |

A.7. For the system to remain bounded, the total mechanical energy of the system must be negative:

$$E' = \frac{1}{2}\mu' v'^2 - \frac{GM_1' M_2}{a} < 0 \rightarrow v' < \sqrt{\frac{2G(M_1' + M_2)}{a}}$$

For an isotropic explosion, we would have $v' = v = \sqrt{\frac{GM}{a}}$ therefore:

$$\sqrt{\frac{G(M_1 + M_2)}{a}} < \sqrt{\frac{2G(M_1' + M_2)}{a}}$$

and:

$$\frac{M_1 - M_2}{2} < M_1'$$

| A-7 (0.7 pt) |
|---|
| $v'_{\max} = \sqrt{\frac{2G(M_1'+M_2)}{a}}$<br>$M'_{1\min} = \frac{M_1 - M_2}{2}$ |

## B. Analysis of the stability of a star

B-1. Using Newton's law of gravity:

$$g = -\frac{4\pi G \int_0^r r'^2 \rho dr'}{r^2} \overset{\rho \cong \rho_c}{\cong} -\frac{4\pi G \rho_c r}{3}$$

B-1 (0.2 pt)

$g = -\frac{4\pi G \rho_c r}{3}$

B-2. Balance of forces for a differential element of volume with a surface area of $A$ and thickness $\Delta r$ between radii $r$ and $r + \Delta r$ is as follows:

$$\vec{F} = -\frac{GM(\vec{r})\rho}{r^2}\mathrm{A}\,\Delta r - \Delta p A = 0$$

in which $M(r)$ is the mass of the part of the star confined within the radius $r$. As $\Delta r$ is small, we can write:

$$\frac{G\rho}{r^2}\left(\int 4\pi r'^2 \rho(r')dr'\right) = -\frac{dp(r)}{dr} = -K\gamma\rho^{\gamma-1}\frac{d\rho}{dr}$$

Multiplying both sides of the equation by $\frac{r^2}{4\pi G\rho}$ and taking the derivative once again, we get:

$$\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] + \frac{4\pi G r^2}{K\gamma}\rho(r) \ = 0$$

B-2 (0.6 pt)

$h_1(\rho, r) = r^2\rho^{\gamma-2}$

$h_2(r) = \frac{4\pi G r^2}{K\gamma}$

B-3.

$$[\rho_c] = ML^{-3}, \quad [p_c] = ML^{-1}T^{-2}, \quad [G] = M^{-1}L^3T^{-2}$$

$$[G^l p_c^m \rho_c^n] = (M^{-1}L^3T^{-2})^l (ML^{-1}T^{-2})^m (ML^{-3})^n = L$$

$$\begin{cases} -l + n + m = 0 \\ 3l - 3n - m = 1 \\ -2l - 2m = 0 \end{cases} \rightarrow \begin{cases} l = -\frac{1}{2} \\ m = \frac{1}{2} \\ n = -1 \end{cases} \rightarrow \ r_0 = G^{-\frac{1}{2}} p_c^{\frac{1}{2}} \rho_c^{-1}$$

B-3 (0.4 pt)

$r_0 = G^{-\frac{1}{2}} p_c^{\frac{1}{2}} \rho_c^{-1}$

B-4.

$$\frac{K\gamma\rho_c^{\gamma-1}}{4\pi G r_0^2 x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = -\rho_c u(r)$$

$$\frac{K\gamma\rho_c^{\gamma-2}}{4\pi G r_0^2 x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = \frac{\gamma}{4\pi x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = -u$$

$$\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] + \frac{4\pi x^2}{\gamma}u = 0$$

| B-4 (0.3 pt) |
|---|
| $A_1(u,x) = x^2u^{\gamma-2}$ |
| $A_2(x) = \frac{4\pi x^2}{\gamma}$ |

B-5.

$$\gamma = 2 \;\rightarrow\; \frac{d}{dx}\left[x^2\frac{du}{dx}\right] = -2\pi x^2u(x) \rightarrow f''(x) = -2\pi f(x) \rightarrow \; f(x) = \frac{\sin(\sqrt{2\pi}x)}{\sqrt{2\pi}}$$

| B-5 (0.6 pt) |
|---|
| $f(x) = \frac{\sin(\sqrt{2\pi}x)}{\sqrt{2\pi}}$ |

B.6.

$$\frac{d^2u}{dx^2} + \frac{(\gamma-2)}{u}\left(\frac{du}{dx}\right)^2 + \frac{2}{x}\left(\frac{du}{dx}\right) + \frac{4\pi}{\gamma}u^{3-\gamma} = 0$$

$$u'(0) = 0 \;\;, \quad \lim_{x\to 0}\frac{u'(x)}{x} = u''(0)$$

$$u''(0) + 2u''(0) + \frac{4\pi}{\gamma} = 0 \quad \rightarrow \quad \gamma = -\frac{4\pi}{3u''(0)}$$

$$\gamma \sim [1.64, 1.70]$$

| B-6 (0.8 pt) |
|---|
| $\gamma = [1.64, 1.70]$ |

B.7.

$$M(r) = \int_0^{\tilde{r}(r,t)} 4\pi r'^2 \tilde{\rho}(r',t)dr' = \int_0^{r} 4\pi r'^2 \rho(r')dr'$$

$$4\pi r^2 \rho(r) = 4\pi \tilde{r}^2 \tilde{\rho}(\tilde{r},t)\frac{\partial \tilde{r}}{\partial r} \rightarrow \frac{\tilde{\rho}}{\rho} = \frac{r^2}{\tilde{r}^2}\left(\frac{\partial \tilde{r}}{\partial r}\right)^{-1} = (1+\epsilon)^{-3} \cong 1-3\epsilon$$

$$\frac{\tilde{g}}{g} = \frac{\frac{GM}{\tilde{r}^2}}{\frac{GM}{r^2}} = \frac{\frac{1}{\tilde{r}^2}}{\frac{1}{r^2}} = (1+\epsilon)^{-2} \cong 1-2\epsilon$$

B-7 (0.9 pt)

$\tilde{g} \simeq g(1-2\epsilon)$

$\tilde{\rho} \simeq \rho(1-3\epsilon)$

B-8. we have

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = \tilde{\rho}(\tilde{g} - \ddot{\tilde{r}})$$

And

$$\tilde{p} = K\tilde{\rho}^{\gamma}$$

So:

$$\ddot{\tilde{r}} = \tilde{g} - \frac{\left(\frac{\partial \tilde{p}}{\partial \tilde{r}}\right)}{\tilde{\rho}} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}}$$

B-8 (0.6 pt)

$$\frac{d^2\tilde{r}}{dt^2} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}}$$

B.9. Using of the results in B.7 and B.8, we have:

$$\frac{d^2\tilde{r}}{dt^2} = \ddot{\tilde{r}} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}} = g(1-2\epsilon) - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r}\left(\frac{(1-3\epsilon)^{\gamma-1}}{(1+\epsilon)}\right)$$
$$= g(1-2\epsilon) - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r}(1-3(\gamma-1)\epsilon-\epsilon)$$

Equilibrium requires:

$$g - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r} = 0 \Rightarrow K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r} = g$$

therefore:

$$\ddot{\tilde{r}} = r\ddot{\epsilon} = g(1-2\epsilon) - g(1-3(\gamma-1)\epsilon-\epsilon) = g(3\gamma-4)\epsilon$$

and:

$$\ddot{\epsilon} = \frac{g}{r}(3\gamma - 4)\epsilon$$

$$\ddot{\epsilon} = -\frac{4\pi G\rho_c}{3}(3\gamma - 4)\epsilon$$

Stability requires that:

$$3\gamma - 4 > 0 \Rightarrow \gamma > \frac{4}{3}$$

and the angular velocity of the oscillations will be:

$$\omega = \sqrt{\frac{4\pi G\rho_c}{3}(3\gamma - 4)}$$

B-9 (0.6 pt)

$$\ddot{\epsilon} = -\frac{4\pi G\rho_c}{3}(3\gamma - 4)\epsilon$$

$$\gamma_{\mathrm{min}} = \frac{4}{3}$$

$$\omega = \sqrt{\frac{4\pi G\rho_c}{3}(3\gamma - 4)}$$