A: Paul Trap

A-1. Due to the symmetry, on the z-axis the only non-zero component of electric field is in the z-direction. So:

$$\vec{E}(0,0,z) = E_z(0,0,z)\,\hat{z} = \hat{z}\int \frac{dq}{4\pi\epsilon_0}\frac{1}{(R^2+z^2)} \times \frac{z}{(R^2+z^2)^{\frac{1}{2}}}$$

The element dq is equal to $\lambda R d\phi$ where ϕ is the angle with the x-axis. Thus:

$$E(0,0,z) = \hat{z}\int \frac{\lambda R d\phi}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}} = \hat{z}\frac{\lambda R}{2\epsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}}$$

For $z \ll R$ this can be written as:

$$E_z(0,0,z) = \frac{\lambda R}{2\epsilon_0}\frac{z}{R^3} = \frac{\lambda z}{2\epsilon_0 R^2}$$

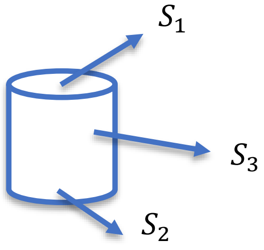
Very close to the z-axis, we can write:

$$E_z(x,y,z) = E_z(0,0,z) + x\frac{\partial E_z}{\partial x}|_{(0,0,z)} + y\frac{\partial E_z}{\partial y}|_{(0,0,z)} + O(x^2,y^2,z^2)$$

Since, there is no difference between x and $-x$ or y and $-y$, it turns out that $\frac{\partial E_z}{\partial x} = \frac{\partial E_z}{\partial y} = 0$. Thus, to the first order in x, y, and z we have:

$$E_z(x,y,z) = \frac{\lambda z}{2\epsilon_0 R^2}$$

Consider a Gaussian surface in the shape of a symmetric cylinder around the z-axis whose bases are parallel with the xy-plane. The cylinder's radius is ρ and its height is $2z$ both of which are small quantities. By Gauss's law we have:



$$0 = \frac{Q_{in}}{\epsilon_0} = \oint \vec{E}.d\vec{S} = \int_{S_1} \vec{E}.d\vec{S} + \int_{S_2} \vec{E}.d\vec{S} + \int_{S_3} \vec{E}.d\vec{S}$$

Integration over S_1 and S_2 gives:

$$\int_{S_1} \vec{E}.d\vec{S} = \int_{S_2} \vec{E}.d\vec{S} = \pi\rho^2 \times \frac{\lambda z}{2\epsilon_0 R^2}.$$

Integration over S_3 involves the ρ-component for which we can write the following expansion:

$$E_\rho(z,\rho,\phi) = E_\rho(0,\rho,\phi) + z\frac{\partial E_\rho}{\partial z}|_{(0,\rho,\phi)} + O(z^2)$$

We have $0 = \frac{\partial E_\rho}{\partial z}|_{(0,\rho,\phi)}$ due to symmetry between z and $-z$, hence, $E_\rho(z,\rho,\phi) = E_\rho(0,\rho,\phi)$ up to the first order. Axial symmetry also implies $\frac{dE_\rho}{d\phi} = 0$. Consequently:

$$\int_{S_3} \vec{E}.d\vec{S} = E_\rho(0,\rho,0) \times 2z \times 2\pi\rho$$

So, Gauss's law implies:

$$0 = E_\rho \times 4\pi z\rho + 2\pi\rho^2 \frac{\lambda z}{2\epsilon_0 R^2}$$

Therefore, E_ρ will be:

$$E_\rho = -\frac{\lambda\rho}{4\epsilon_0 R^2}$$

In the cylindrical coordinate we will have:

$$\vec{E}(\rho,\phi,z) = -\frac{\lambda\rho}{4\epsilon_0 R^2}\hat{\rho} + \frac{\lambda z}{2\epsilon_0 R^2}\hat{z}$$

In cartesian coordinates we will have:

$$\vec{E}(x,y,z) = \frac{\lambda}{4\epsilon_0 R^2}(-x,-y,2z)$$

Since the ring is positively charged, the equilibrium in the x and y directions are stable, while the equilibrium in the z-direction is unstable. The equations of motion in the x and y directions read:

$$m\ddot{x} = qE_x = -\frac{q\lambda}{4\epsilon_0 R^2}x$$

$$m\ddot{y} = qE_y = -\frac{q\lambda}{4\epsilon_0 R^2}y$$

Therefore, the frequencies of small oscillations are:

$$\omega_x^2 = \omega_y^2 = \frac{q\lambda}{4\epsilon_0 R^2 m}$$

A-1 (1.5 pt)

(a) $\vec{E}(x, y, z) = \frac{-\lambda x}{4\epsilon_0 R^2}\hat{x} + \frac{-\lambda y}{4\epsilon_0 R^2}\hat{y} + \frac{\lambda z}{2\epsilon_0 R^2}\hat{z}$

(b) $\omega_x = \omega_y = \sqrt{\frac{Q\lambda}{4\epsilon_0 R^2 m}}$

A-2.

The force in the z-direction is:

$$F_z = qE_z = \frac{Q\lambda z}{2\epsilon_0 R^2} = \frac{Q}{2\epsilon_0 R^2}\lambda_0 z + \frac{Qu}{2\epsilon_0 R^2}\cos\Omega t\, z$$

the equation of motion can thus be written as:

$$\ddot{z} = \left(\frac{Q\lambda_0}{2\epsilon_0 R^2 m} + \frac{Qu}{2\epsilon_0 R^2 m}\cos\Omega t\right) z$$

Therefore:

$$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}} \quad , \quad a = \frac{Qu}{2\epsilon_0 R^2 m\Omega^2}$$

A-2 (0.4 pt)

$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}}$, $a = \frac{Qu}{2\epsilon_0 R^2 m\Omega^2}$

A.3.

$$z = p(t) + q(t) \quad \rightarrow \quad \ddot{p} + \ddot{q} = (k^2 + a\Omega^2\cos\Omega t)(p + q)$$

1. We are assuming that p is almost constant, $\ddot{p} \simeq 0$.
2. According to the assumptions $k^2 \ll a\Omega^2$ and $q \ll p$ we can ignore k^2 in the first term on the right-hand side of the equation and q in the second term.

hence, the equation of motion can be simplified as follows:

$$\ddot{q} = pa\Omega^2\cos\Omega t.$$

As we have assumed that p is a constant, the second derivative of q is just proportional to $\cos\Omega t$ which gives:

$$q = -pa\cos\Omega t + c_1 t + c_2.$$

Since q is supposed to remain small, c_1 must vanish. Also $c_2 = 0$ because the mean value of q is supposed to remain zero. Therefore:

$$q = -pa\cos\Omega t$$

A-3 (1.8 pt)
(a) $\ddot{q}(t) = pa\Omega^2\cos\Omega t$
(b) $q(t) = -pa\cos\Omega t$

A-4. Using the final result for q the equation of motion for p reads:

$$\ddot{p} + pa\Omega^2\cos\Omega t = (k^2 + a\Omega^2\cos\Omega t)(p - ap\cos\Omega t)$$

Which gives:

$$\ddot{p} = k^2 p - ak^2 p\cos\Omega t - a^2\Omega^2 p\cos^2\Omega t$$

Averaging over one period, we'll have:

$$\langle\cos\Omega t\rangle = 0 \quad , \quad \langle\cos^2\Omega t\rangle = \frac{1}{2}$$

and:

$$\ddot{p} = \left(k^2 - \frac{a^2\Omega^2}{2}\right)p.$$

In order for the motion to be stable, the expression inside the parentheses should be negative, i.e.

$$\frac{a^2\Omega^2}{2} > k^2 \quad \rightarrow \quad \Omega > \sqrt{2}\frac{k}{a}$$

A-4 (1.5 pt)
(a) $\ddot{p}(t) = \left(k^2 - \frac{a^2\Omega^2}{2}\right)p$
(b) $\Omega > \sqrt{2}\frac{k}{a}$

A.5. With the given data we have:

$$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}} = 2\times 10^5\ \mathrm{rad/s}$$

$$a = 0.04 \quad \rightarrow \quad \Omega_{\min} = 7\times 10^6\ \mathrm{rad/s}$$

which is in the range of radio waves.

A-5 (0.4 pt)
$k = 2\times 10^5\ \mathrm{rad/s}$
$\Omega_{\min} = 7\times 10^6 \mathrm{rad/s}$

B: Doppler Cooling

B-1. From the uncertainty principle we know:

$$\Delta E \times \Delta t \simeq \hbar$$

Here Δt is the time τ and $\Delta E = \hbar\Delta\omega$. So:

$$\hbar\Delta\omega \times \tau \simeq \hbar \quad \rightarrow \quad \Delta\omega \simeq \frac{1}{\tau} = \Gamma$$

B-1 (0.5 pt)
$\Gamma = \frac{1}{\tau}$

B-2. We denote the forward and backward collision rates by s_+ and s_- respectively. Let us proceed in the atom's frame of reference. Ignoring the terms of the order $\frac{v^2}{c^2}$, the Doppler effect can be written in the following form:

$$\omega' = \omega\left(1 + \frac{v}{c}\right)$$

Taking the atom's velocity in the positive x-direction, we have:

$$\omega_+ = \omega_{\mathrm{L}}\left(1 + \frac{v}{c}\right)$$

$$\omega_- = \omega_{\mathrm{L}}\left(1 - \frac{v}{c}\right)$$

So:

$$s_+ = s_{\mathrm{L}} + \alpha\left(\omega_{\mathrm{L}}\left(1 + \frac{v}{c}\right) - \omega_{\mathrm{L}}\right) = s_{\mathrm{L}} + \alpha\omega_{\mathrm{L}}\frac{v}{c}$$

$$s_- = s_L + \alpha\left(\omega_{\mathrm{L}}\left(1 - \frac{v}{c}\right) - \omega_{\mathrm{L}}\right) = s_{\mathrm{L}} - \alpha\omega_{\mathrm{L}}\frac{v}{c}$$

The momentum transfer per unit time from the oncoming photons to the atom is equal to:

$$\pi_+ = s_+ \times (-\hbar k_+)$$

For the backward photons we have:

$$\pi_- = s_- \times (+\hbar k_-)$$

Where $k_\pm = \frac{\hbar\omega_\pm}{c}$.

The total momentum transferred to the atom per unit time is equal to:

$$\pi_+ + \pi_- = -2\hbar k_{\mathrm{L}}\frac{v}{c}\,\omega_{\mathrm{L}}\alpha\left(1 + \frac{s_{\mathrm{L}}}{\alpha\omega_{\mathrm{L}}}\right)$$

Where with the approximation $s_{\mathrm{L}} \ll \alpha\omega_{\mathrm{L}}$, we will arrive at:

$$\pi_+ + \pi_- = -2\hbar k_{\mathrm{L}}\frac{v}{c}\,\omega_{\mathrm{L}}\alpha$$

Note that as the atom is heavy, its velocity almost doesn't change after the absorption of the photon. Therefore, there will be almost no Doppler shifting in the re-emitted photon and hence, on average there will be no momentum transfer to the atom during the re-emission process.

The above expression is, in fact, the force. Since $v > 0$, we have:

$$F = -(2\alpha\hbar k_{\mathrm{L}}^2)v$$

The same result holds for $v < 0$. This is in the atom's reference frame. However, as we have kept only up to the first order in v/c, the same result holds in the lab frame:

$$F = -(2\alpha\hbar k_{\mathrm{L}}^2)v$$

B-2 (1.7 pt)

$$s_+ = s_L + \alpha\omega_L \frac{v}{c}$$
$$s_- = s_L - \alpha\omega_L \frac{v}{c}$$

$$\pi_+ = s_+ \times (-\hbar k_+)$$

$$\pi_- = s_- \times (+\hbar k_-)$$

$$F = -(2\alpha\hbar k_L^2)v$$

B-3. The atom's momentum before the collision is zero. After the collision it will be (assuming the photon's momentum is in the x-direction):

$$P_1 = \hbar k_L$$

After re-emitting the photon, we may have two equally likely outcomes for the final momentum:

1. The photon is emitted in the positive x-direction which causes the atom's momentum to become zero
2. The photon is emitted in the negative x-direction which causes the atom's momentum to become: $P_f = +2\hbar k_L$

Thus, the mean final energy is equal to:

$$\langle E_f \rangle = \langle \frac{P_f^2}{2m} \rangle = \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{4\hbar^2 k_L^2}{2m} = \frac{\hbar^2 k_L^2}{m}$$

This process occurs during the time τ. So, the input power (the power gained by the atom as a result of this process) is equal to:

$$P_{in} = \frac{\hbar^2 k_L^2}{m\tau}$$

B-3 (1.0 pt)

$$P_{in} = \frac{\hbar^2 k_L^2}{m\tau}$$

B.4. The output power (the power lost by the atom through collision with laser photons) can be written as:

$$P_{\text{out}} = F \cdot v = -2\alpha\hbar k_{\text{L}}^2 v^2$$

At equilibrium we should have:

$$P_{\text{out}} + P_{\text{in}} = 0 \quad \rightarrow \quad \frac{\hbar^2 k_{\text{L}}^2}{m\tau} = 2\alpha\hbar k_{\text{L}}^2 \overline{v^2} \quad \rightarrow \quad \overline{v^2} = \frac{\hbar\Gamma}{2\alpha m}$$

And the temperature of this system is equal to:

$$\frac{1}{2} m\overline{v^2} = \frac{1}{2} k_{\text{B}} T \quad \rightarrow \quad T = \frac{\hbar\Gamma}{2\alpha k_{\text{B}}}$$

B-4 (0.8 pt)

$P_{\text{out}} = -2\alpha\hbar k_{\text{L}}^2 v^2$

$\overline{v^2} = \frac{\hbar\Gamma}{2\alpha m}$

$T = \frac{\hbar\Gamma}{2\alpha k_{\text{B}}}$

B-5. Considering the given data:

$$T = \frac{1\,055 \times 10^{\ 34}\ \text{J.s}}{2 \times 4 \times 1\,381 \times 10^{\ 23}\ \text{J/K} \times 5 \times 10^{\ 9}\ \text{s}} = 2 \times 10^{-4}\ \text{K}$$

B-5 (0.4 pt)

$T = 2 \times 10^{-4}$ K