



## Electrostatic lens (10 points)

### Part A. Electrostatic potential on the axis of the ring (1 point)

#### A.1 (0.3 points)

The linear charge density of the ring is  $\lambda = q/(2\pi R)$ . All the points of the ring are situated a distance  $\sqrt{R^2 + z^2}$  away from point A. Integrating over the whole ring we readily obtain:

$$\Phi(z) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 + z^2}}.$$

#### A.2 (0.4 points)

Using an expansion in powers of  $z$  we obtain:

$$\Phi(z) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 + z^2}} = \frac{q}{4\pi\epsilon_0 R} \frac{1}{\sqrt{1 + \left(\frac{z}{R}\right)^2}} \approx \frac{q}{4\pi\epsilon_0 R} \left(1 - \frac{z^2}{2R^2}\right).$$

#### A.3 (0.2 points)

The potential energy of the electron is  $V(z) = -e\Phi(z)$ . The force acting on the electron is

$$F(z) = -\frac{dV(z)}{dz} = +e \frac{d\Phi}{dz} = -\frac{qe}{4\pi\epsilon_0 R^3} z.$$

If this is a restoring force, it should be negative for positive  $z$ . Thus,  $q > 0$ .

#### A.4 (0.1 points)

The equation of motion for an electron is

$$m\ddot{z} + \frac{qe}{4\pi\epsilon_0 R^3} z = 0$$

(here dots denote time derivatives). We therefore get

$$\omega = \sqrt{\frac{qe}{4\pi m\epsilon_0 R^3}}.$$

**Part B. Electrostatic potential in the plane of the ring (1.7 points)****B.1 (1.5 points)**

There are two different ways to solve this problem: (i) using direct integration; (ii) using Gauss's law and the result of part A.

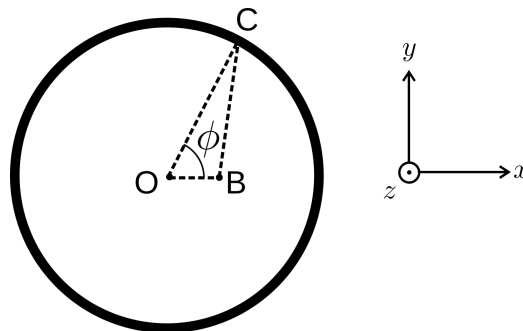


Figure 1: Calculating electrostatic potential in the plane of the ring through direct integration.

(i) **Direct integration.** We will follow the notations of Figure 1. Since the potential has cylindrical symmetry, let the point B, where we calculate the potential, be on the x-axis. Let

$$|OB| = r; |OC| = R.$$

Thus:

$$|BC|^2 = R^2 + r^2 - 2Rr \cos \phi.$$

Electrostatic potential created by ring element  $d\phi$  at the point B:

$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{\lambda R d\phi}{\sqrt{R^2 + r^2 - 2Rr \cos \phi}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda d\phi}{\sqrt{1 + \frac{r^2}{R^2} - 2\frac{r}{R} \cos \phi}}.$$

Using the expansion given in the formulation of the problem for  $\epsilon = -1/2$  we have:

$$d\Phi \approx \frac{\lambda d\phi}{4\pi\epsilon_0} \left[ 1 - \frac{1}{2} \left( \frac{r^2}{R^2} - 2\frac{r}{R} \cos \phi \right) + \frac{3}{8} \left( \frac{r^2}{R^2} - 2\frac{r}{R} \cos \phi \right)^2 \right].$$

Ignoring the terms of the order  $r^3$  and  $r^4$  we get:

$$d\Phi \approx \frac{\lambda d\phi}{4\pi\epsilon_0} \left[ 1 + \frac{r}{R} \cos \phi + \frac{r^2}{R^2} \left( \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right) \right].$$

Integrating over all angles we finally obtain:

$$\Phi(r) = \frac{\lambda}{4\pi\epsilon_0} \int_0^{2\pi} \left[ 1 + \frac{r}{R} \cos \phi + \frac{r^2}{R^2} \left( \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right) \right] d\phi.$$



$$\Phi(r) = \frac{q}{4\pi\epsilon_0 R} \left( 1 + \frac{r^2}{4R^2} \right).$$

From here, comparing with the expression  $\Phi(r) = q(\alpha + \beta r^2)$ , we obtain

$$\beta = \frac{1}{16\pi\epsilon_0 R^3}.$$

(ii) Gauss's law.

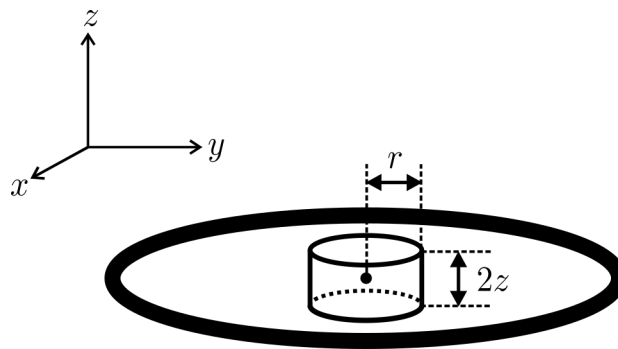


Figure 2: Calculating electrostatic potential in the plane of the ring via Gauss's law.

Let us analyze a small cylinder of radius  $r$ . The center of the cylinder coincides with the center of the ring. In part A we analyzed the potential along the  $z$ -axis, while in this part we analyze the potential along the radius  $r$ . For any  $z \ll R$  and  $r \ll R$  the potential has an expression:

$$\Phi(z, r) = \frac{q}{4\pi\epsilon_0 R} \left( 1 - \frac{z^2}{2R^2} \right) + q\beta r^2.$$

The lowest order terms are quadratic in  $r$  and  $z$ . Due to reflection symmetry the potential does not contain terms of the type  $rz$ . This, for example, immediately gives us  $\alpha = 1/(4\pi\epsilon_0 R)$ . Thus, for small  $r$  and  $z$  electric fields in the radial and axial directions are:

$$\mathcal{E}_z(z, r) = +\frac{q}{4\pi\epsilon_0 R^3} z, \quad \mathcal{E}_r(z, r) = -2q\beta r.$$

Applying Gauss's law to the cylinder we obtain:

$$\oint \vec{\mathcal{E}} \cdot d\vec{S} = 0 \quad \Rightarrow \quad \int_{\text{side}} \vec{\mathcal{E}} \cdot d\vec{S} + \int_{\text{base}} \vec{\mathcal{E}} \cdot d\vec{S} = 0.$$

The second integral is:

$$\int_{\text{base}} \vec{\mathcal{E}} \cdot d\vec{S} = 2\pi r^2 \mathcal{E}_z(z, r) = \frac{qzr^2}{2\epsilon_0 R^3}.$$

The first integral is:

$$\int_{\text{side}} \vec{\mathcal{E}} \cdot d\vec{S} = 4\pi r z \mathcal{E}_r(z, r) = -8\pi q\beta r^2 z.$$

Gauss's theorem thus gives:

$$\frac{qzr^2}{2\epsilon_0 R^3} - 8\pi q\beta r^2 z = 0.$$

This immediately yields

$$\beta = \frac{1}{16\pi\epsilon_0 R^3},$$

which agrees with the result obtained via direct integration.

## B.2 (0.2 points)

The potential of the electron is  $V(r) = -e\Phi(r)$ . Force acting on the electron in the  $xy$  plane is

$$F(r) = -\frac{dV(r)}{dr} = +e\frac{d\Phi(r)}{dr} = \frac{qe}{8\pi\epsilon_0 R^3}r.$$

To have oscillations we need the force to be negative for  $r > 0$ . Thus,  $q < 0$ .

## Part C. The focal length of the idealized electrostatic lens (2.3 points)

### C.1 (1.3 points)

Let us consider an electron with the velocity  $v = \sqrt{2E/m}$  at a distance  $r$  from the “optical” axis (Figure 2 of the problem). The electron crosses the “active region” of the lens in time

$$t = \frac{d}{v}.$$

The equation of motion in the  $r$  direction:

$$m\ddot{r} = 2eq\beta r.$$

During the time the electron crosses the active region of the lens, the electron acquires radial velocity:

$$v_r = \frac{2eq\beta r}{m} \frac{d}{v} < 0.$$

The lens will be focusing if  $q < 0$ . The time it takes for an electron to reach the “optical” axis is:

$$t' = \frac{r}{|v_r|} = -\frac{mv}{2eq\beta d}.$$

During this time the electron travels in the  $z$ -direction a distance

$$\Delta z = t'v = -\frac{mv^2}{2eq\beta d} = -\frac{E}{eqd\beta}.$$

$\Delta z$  does not depend on the radial distance  $r$ , therefore all electron will cross the “optical” axis (will be focused) in the same spot. Thus,

$$f = -\frac{E}{eqd\beta}.$$



C.2 (0.8 points)

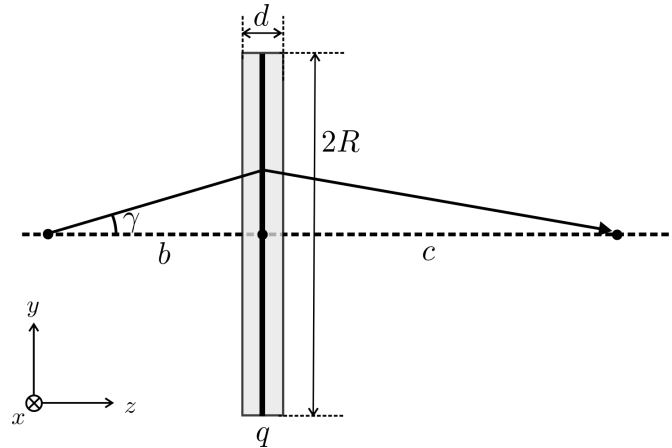


Figure 3: Focusing of electrons.

Let us consider an electron emitted at an angle  $\gamma$  to the optical axis (Figure 3). Its initial velocity in the radial direction is:

$$v_{r;0} = v \sin \gamma \approx v \gamma \approx v \frac{r}{b},$$

where  $r$  is the radial distance of the electron when it reaches the plane of the ring. The velocity in the  $z$ -direction is

$$v_z = v \cos \gamma \approx v.$$

For small angles  $\gamma$  the additional velocity in the  $r$ -direction acquired in the “active region” is the same as in part C.1. Thus, the radial velocity after crossing the active region is

$$v_r = v \frac{r}{b} + \frac{2eq\beta r d}{m} \frac{1}{v},$$

where the first term is positive and the second term is negative, since  $q < 0$ . If the electrons are focused, then  $v_r < 0$  (this can be verified after obtaining the final result). The electron will reach the optical axis in time

$$t' = \frac{r}{|v_r|} = -\frac{r}{\frac{2eq\beta r d}{m} \frac{1}{v} + v \frac{r}{b}} = -\frac{1}{\frac{2eq\beta d}{m} \frac{1}{v} + \frac{v}{b}}.$$

During this time it will travel a distance

$$c = t'v = -\frac{1}{\frac{2eq\beta d}{m} \frac{1}{v^2} + \frac{1}{b}} = -\frac{1}{\frac{eq\beta d}{E} + \frac{1}{b}}.$$

C.3 (0.2 pt)

From the previous answer we obtain:

$$\frac{1}{b} + \frac{1}{c} = -\frac{eq\beta d}{E}.$$



Comparing with the answer of C.1 we immediately obtain

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{f},$$

i.e. the equation of a thin optical lens is valid for an electrostatic lens as well.

### Part D. The ring as a capacitor (3 points)

#### D.1 (2.0 points)

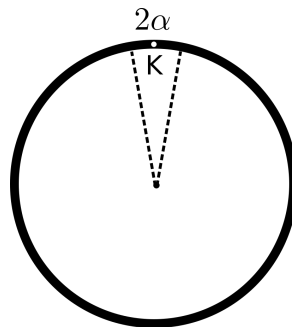


Figure 4: Calculation of the capacitance of the ring.

Let us sub-divide the entire ring into two parts: a part corresponding to the angle  $2\alpha \ll 1$ , and the rest of the ring, as shown in Figure 4. While the angle is small in comparison to 1, let us assume that the length of the first part,  $\alpha R$ , is still large compared to  $a$  ( $\alpha R \gg a$ ). Let us calculate the electrostatic potential  $\Phi$  at point K. It is a sum of two terms: the first one produced by the cut-out part with an angle  $2\alpha$  (contribution  $\Phi_1$ ) and the second one originating from the rest of the ring (contribution  $\Phi_2$ ).

Contribution  $\Phi_1$ . Since  $\alpha \ll 1$ , we can neglect the curvature of the cylinder that is cut out from the ring. The linear charge density on the ring is  $\lambda = \frac{q}{2\pi R}$ . The potential at the center of the cylinder is then given by an integral:

$$\Phi_1 = 2 \frac{1}{4\pi\epsilon_0} \frac{q}{2\pi R} \int_0^{\alpha R} \frac{dx}{\sqrt{x^2 + a^2}} = \frac{q}{4\pi^2\epsilon_0 R} \int_0^{\alpha R} \frac{d(x/a)}{\sqrt{1 + (x/a)^2}} = \frac{q}{4\pi^2\epsilon_0 R} \int_0^{\alpha R/a} \frac{dy}{\sqrt{1 + y^2}}.$$

Using the integral provided in the description of the problem we get:

$$\Phi_1 = \frac{q}{4\pi^2\epsilon_0 R} \ln \left( y + \sqrt{1 + y^2} \right) \Big|_0^{\alpha R/a} = \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{\alpha R}{a} + \sqrt{1 + \left( \frac{\alpha R}{a} \right)^2} \right).$$

As  $\alpha R \gg a$ ,

$$\Phi_1 \approx \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{2\alpha R}{a} \right).$$

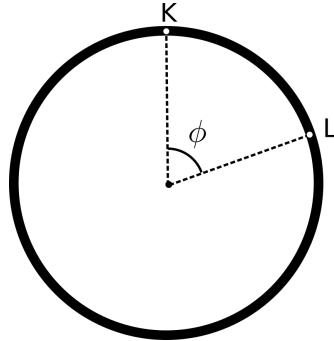


Figure 5: Calculation of the capacitance of the ring

Contribution  $\Phi_2$ . In this case we can neglect the thickness  $a$ . Using the cosine theorem we can derive the distance between points K and L of Figure 5:

$$|KL| = 2R \sin \frac{\phi}{2}.$$

The contribution  $\Phi_2$  can then be written as an integral:

$$\Phi_2 = 2 \frac{q}{2\pi} \frac{1}{4\pi\epsilon_0} \int_{\alpha}^{\pi} \frac{d\phi}{2R \sin \frac{\phi}{2}} = \frac{q}{8\pi^2\epsilon_0 R} \int_{\alpha}^{\pi} \frac{d\phi}{\sin \frac{\phi}{2}} = \frac{q}{4\pi^2\epsilon_0 R} \int_{\alpha}^{\pi} \frac{d\left(\frac{\phi}{2}\right)}{\sin \frac{\phi}{2}} = \frac{q}{4\pi^2\epsilon_0 R} \int_{\alpha/2}^{\pi/2} \frac{d\chi}{\sin \chi}.$$

Using the integral from the formulation of the problem, we calculate:

$$\int_{\alpha/2}^{\pi/2} \frac{d\chi}{\sin \chi} = -\ln \left( \frac{\cos \chi + 1}{\sin \chi} \right) \Big|_{\alpha/2}^{\pi/2} = \ln \left( \frac{\cos \alpha/2 + 1}{\sin \alpha/2} \right) \approx \ln \left( \frac{4}{\alpha} \right)$$

for  $\alpha \ll 1$ . Therefore

$$\Phi_2 \approx \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{4}{\alpha} \right).$$

The total potential and capacitance. The total potential is the sum of  $\Phi_1$  and  $\Phi_2$ :

$$\Phi = \Phi_1 + \Phi_2 = \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{2\alpha R}{a} \right) + \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{4}{\alpha} \right) = \frac{q}{4\pi^2\epsilon_0 R} \ln \left( \frac{8R}{a} \right).$$

$\alpha$  drops out from the expression. From here we obtain the capacitance  $C = q/\Phi$ :

$$C = \frac{4\pi^2\epsilon_0 R}{\ln \left( \frac{8R}{a} \right)}.$$

$C \rightarrow 0$  as  $a \rightarrow 0$ .



**D.2 (1.0 point)**

Let  $q(t)$  be the charge on the ring at a time  $t$ . Potential of the disk is thus  $q(t)/C$ . Voltage drop of the resistor is  $R_0 I(t) = R_0 dq/dt$ . Therefore for time  $-\frac{d}{2v} < t < \frac{d}{2v}$ :

$$\frac{q(t)}{C} + R_0 \frac{dq}{dt} = V_0.$$

Integrating this equation and keeping in mind that  $q(t) = 0$  at  $t = -d/(2v)$ , we get:

$$q(t) = CV_0 \left( 1 - e^{-\frac{d}{2vR_0C}} e^{-\frac{t}{R_0C}} \right).$$

The charge attains the largest absolute value at  $t = d/(2v)$ . The value of the charge at this time is:

$$q_0 = CV_0 \left( 1 - e^{-\frac{d}{vR_0C}} \right).$$

When  $t > \frac{d}{2v}$ , we get:

$$\frac{q(t)}{C} + R_0 \frac{dq}{dt} = 0.$$

From here:

$$q(t) = q_0 e^{-\frac{t}{R_0C} + \frac{d}{2vR_0C}} = CV_0 \left( e^{\frac{d}{2vR_0C}} - e^{-\frac{d}{2vR_0C}} \right) e^{-\frac{t}{R_0C}}.$$

Therefore, we obtain:

$$q(t) = \begin{cases} 0 & \text{for } t < -\frac{d}{2v}; \\ CV_0 \left( 1 - e^{-\frac{d}{2vR_0C}} e^{-\frac{t}{R_0C}} \right) & \text{for } -\frac{d}{2v} < t < \frac{d}{2v}; \\ CV_0 \left( e^{\frac{d}{2vR_0C}} - e^{-\frac{d}{2vR_0C}} \right) e^{-\frac{t}{R_0C}} & \text{for } t > \frac{d}{2v}. \end{cases}$$

For a lens to be focusing we require that charge is negative, therefore  $V_0 < 0$ . The dependence of charge on time is shown in Figure 6.

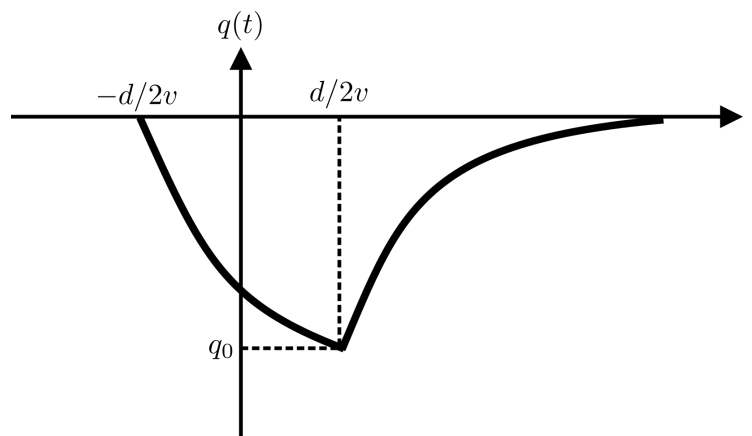


Figure 6: Charge on the ring as a function of time.



**Part E. Focal length of a more realistic lens (2 points)**

**E.1 (1.7 points)**

Like in part C, the radial equation of motion of an electron is:

$$m\ddot{r} = 2eq(t)\beta r,$$

where in this case  $q(t)$  depends on time. Using the notation  $\eta = 2e\beta/m$ , we obtain:

$$\ddot{r} - \eta q(t)r = 0.$$

As  $f/v \gg R_0C$ , then during charging–decharging the electron does not substantially change its radial position  $r$ , and we can assume  $r$  to be constant during the entire charging–decharging process. In this case the acquired vertical velocity is

$$v_r = \eta r \int_{-d/(2v)}^{\infty} q(t) dt.$$

We can use the derived equations for  $q(t)$  and find the integrals. The integral  $\int_{-d/(2v)}^{d/(2v)} q(t) dt$  is (using the notation  $d/v = t_0$ ,  $R_0C = \tau$ ,  $CV_0 = Q_0$ ):

$$\int_{-t_0/2}^{t_0/2} q(t) dt = \int_{-t_0/2}^{t_0/2} Q_0 \left( 1 - e^{-\frac{t_0}{2\tau}} e^{-\frac{t}{\tau}} \right) dt = Q_0 \left( t_0 - \tau \left[ 1 - e^{-t_0/\tau} \right] \right).$$

The integral  $\int_{d/(2v)}^{\infty} q(t) dt$  is

$$\int_{t_0/2}^{\infty} Q_0 \left( e^{\frac{t_0}{2\tau}} - e^{-\frac{t_0}{2\tau}} \right) e^{-\frac{t}{\tau}} dt = Q_0 \tau \left[ 1 - e^{-t_0/\tau} \right].$$

Adding the two integrals we obtain for the final integral:

$$\int_{-t_0/2}^{\infty} q(t) dt = Q_0 t_0.$$

Interestingly, it does not depend on  $\tau = R_0C$ . Therefore, the acquired vertical velocity of the electron is

$$v_r = \eta r \frac{CV_0 d}{v} = \frac{2e\beta CV_0 d r}{mv}.$$

Following the logic similar to part C, we derive the focal length

$$f = -\frac{E}{eCV_0 d \beta}.$$

**E.2 (0.3 points).**

Comparing  $f = -E/(eCV_0 d \beta)$  with  $f = -E/(eqd\beta)$  from part C we immediately obtain  $q_{\text{eff}} = CV_0$ .