Dark Matter

A. Cluster of Galaxies

Question A.1

Answer	Marks
Potential energy for a system of a spherical object with mass $M(r)=\frac{4}{3}\pi r^3\rho$ and a test particle with mass dm at a distance r is given by $$dU=-G\frac{M(r)}{r}dm$$	0.2 pts
Thus for a sphere of radius R $$U=-\int_0^R G\frac{M(r)}{r}dm=-\int_0^R G\frac{4\pi r^3\rho}{3r}4\pi r^2\rho dr=-\frac{16}{3}G\pi^2\rho^2\int_0^R r^4 dr$$ $$=-\frac{16}{15}G\pi^2\rho^2 R^5$$	0.6 pts
Then using the total mass of the system $$M=\frac{4}{3}\pi R^3\rho$$ we have $$U=-\frac{3}{5}\frac{GM^2}{R}$$	0.2 pts
Total	1.0 pts

Question A.2

Answer	Marks
Using the Doppler Effect, $$f_i = f_0 \frac{1}{1+\beta} \approx f_0(1-\beta),$$ where $\beta = v/c$ and $v << c$. Thus the i-th galaxy moving away (radial) speed is $$V_{ri} = -\frac{f_i - f_0}{f_0} c$$ Alternative without approximation: $$f_i = f_0 \frac{1}{1+\beta}$$ $$V_{ri} = c\left(\frac{f_0}{f_i} - 1\right)$$	0.2 pts
All the galaxies in the galaxy cluster will be moving away together due to the cosmological expansion. Thus the average moving away speed of the N galaxies in the cluster is $$V_{cr} = -\frac{c}{Nf_0}\sum_{i=1}^{N}(f_i - f_0) = -\frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0} - 1\right).$$ Alternative without approximation: $$V_{cr} = \frac{cf_0}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i} - \frac{1}{f_0}\right) = \frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i} - 1\right)$$	0.3 pts
Total	0.5 pts

Question A.3

Answer	Marks
The galaxy moving away speed V_i, in part A.2, is only one component of the three component of the galaxy velocity. Thus the average square speed of each galaxy with respect to the center of the cluster is $$\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{1}{N}\sum_{i=1}^{N}(V_{xi}-V_{xc})^2+(V_{yi}-V_{yc})^2+(V_{zi}-V_{zc})^2$$ Due to isotropic assumption $$\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{cr})^2$$	0.5 pts
And thus the root mean square of the galaxy speed with respect to the cluster center is $$v_{rms}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{rc})^2}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}{}^2-2V_{cr}V_{ri}+V_{cr}{}^2)}=\sqrt{\frac{3}{N}\left(\sum_{i=1}^{N}V_{ri}{}^2\right)-3V_{cr}{}^2}$$ $$v_{rms}=c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)\right)^2}$$ $$=\frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(f_i^2-2f_if_0+f_0^2\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}f_i\right)^2-2\frac{f_0}{N}\sum_{i=1}^{N}f_i+f_0^2\right)}$$ $$=\frac{c\sqrt{3}}{f_0N}\sqrt{\left(N\sum_{i=1}^{N}f_i^2\right)-\left(\sum_{i=1}^{N}f_i\right)^2}$$ Alternative without approximation:	0.7 pts

$v_{rms} = c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)\right)^2}$ $= \frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i^2}-2\frac{1}{f_i}\frac{1}{f_0}+\frac{1}{f_0^2}\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{f_i}\right)^2-2\frac{1}{N}\frac{1}{f_0}\sum_{i=1}^{N}\frac{1}{f_i}+\frac{1}{f_0^2}\right)}$ $= \frac{cf_0\sqrt{3}}{N}\sqrt{\left(N\sum_{i=1}^{N}\left(\frac{1}{f_i}\right)^2\right)-\left(\sum_{i=1}^{N}\frac{1}{f_i}\right)^2}$	
The mean kinetic energy of the galaxies with respect to the center of the cluster is $K_{ave} = \frac{m}{2}\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2 = \frac{m}{2}v_{rms}^{\ 2}$	0.3 pts
Total	1.5 pts

Question A.4

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The time average of $d\Gamma / dt$ vanishes

$$\left\langle \frac{d\Gamma}{dt} \right\rangle_t = 0$$

Now

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i \vec{v}_i \cdot \vec{v}_i = \sum_i \vec{F}_i \cdot \vec{r}_i + 2K$$

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Where K is the total kinetic energy of the system. Since the gravitational force on i-th particle comes from its interaction with other particles then

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i>j} \vec{F}_{ij} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_j$$

$$= \sum_{i<j} \vec{F}_{ji} \cdot (\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|^2} \frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|} \cdot (\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|} = U_{\text{tot}}$$

Alternative proof:

$$\sum_i \vec{F}_i . \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} . \vec{r}_i = \vec{F}_{21} . \vec{r}_1 + \vec{F}_{31} . \vec{r}_1 + \vec{F}_{41} . \vec{r}_1 + \cdots + \vec{F}_{N1} . \vec{r}_1 +$$

$$\vec{F}_{12} . \vec{r}_2 + \vec{F}_{32} . \vec{r}_2 + \vec{F}_{42} . \vec{r}_2 + \cdots + \vec{F}_{N2} . \vec{r}_2 +$$

$$\vec{F}_{13} . \vec{r}_3 + \vec{F}_{23} . \vec{r}_3 + \vec{F}_{43} . \vec{r}_3 + \cdots + \vec{F}_{N3} . \vec{r}_3 + \ldots$$

$$\vec{F}_{1N} . \vec{r}_N + \vec{F}_{2N} . N_N + \vec{F}_{3N} . \vec{r}_N + \cdots + \vec{F}_{NN-1} . \vec{r}_{N-1}$$

Collecting terms and noting that $\vec{F}_{ij} = -\vec{F}_{ji}$ we have

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$$\vec{F}_{12}.(\vec{r}_2-\vec{r}_1)+\vec{F}_{13}.(\vec{r}_3-\vec{r}_1)+\vec{F}_{14}.(\vec{r}_4-\vec{r}_1)+\cdots+\vec{F}_{23}.(\vec{r}_3-\vec{r}_2)+\vec{F}_{24}.(\vec{r}_4-\vec{r}_2)+\cdots+\vec{F}_{34}.(\vec{r}_4-\vec{r}_3)+\cdots=\sum_{i<j}\vec{F}_{ji}.(\vec{r}_i-\vec{r}_j)=-\sum_{i<j}G\frac{m_i m_j}{\lvert\vec{r}_i-\vec{r}_j\rvert^2}\frac{(\vec{r}_i-\vec{r}_j)}{\lvert\vec{r}_i-\vec{r}_j\rvert}.(\vec{r}_i-\vec{r}_j)=-\sum_{i<j}G\frac{m_i m_j}{\lvert\vec{r}_i-\vec{r}_j\rvert}=U_{tot}$$	
Thus we have $$\frac{d\Gamma}{dt}=U+2K$$ And by taking its time average we obtain $\left\langle\frac{d\Gamma}{dt}=U+2K\right\rangle_t=0$ and thus $\langle K\rangle_t=-\frac{1}{2}\langle U\rangle_t$. Therefore $\gamma=\frac{1}{2}$.	0.2 pts
Total	1.7 pts

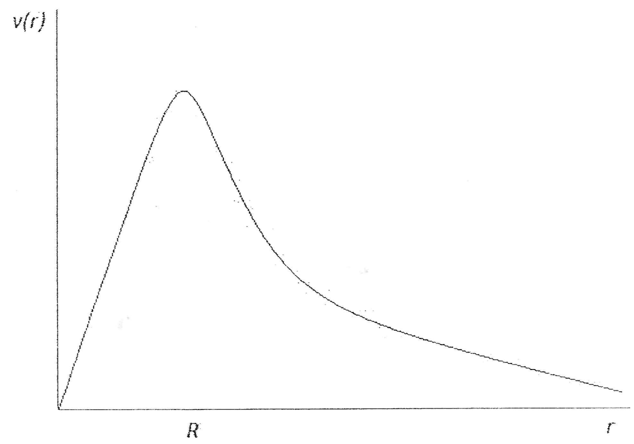
Question A.5

Answer	Marks
Using Virial theorem, and since the dark matter has the same root mean square speed as the galaxy, then we have $\langle K \rangle_t = -\frac{1}{2}\langle U \rangle_t$ $\frac{M}{2}v_{rms}^2 = \frac{1}{2}\frac{3}{5}\frac{GM^2}{R}$	0.3 pts
From which we have $M = \frac{5Rv_{rms}^2}{3G}$	0.1 pts
And the dark matter mass is then $M_{dm} = \frac{5Rv_{rms}^2}{3G} - Nm_g$	0.1 pts
Total	0.5 pts

B. Dark Matter in a Galaxy

Question B.1

Answer	Marks
Answer B.1: The gravitational attraction for a particle at a distance *r* from the center of the sphere comes only from particles inside a spherical volume of radius *r*. For particle inside the sphere with mass m_s , assuming the particle is orbiting the center of mass in a circular orbit, we have $$G\frac{m'(r)m_s}{r^2}=\frac{m_s v_0^2}{r}$$	0.3 pts
with $m'(r)$ is the total mass inside a sphere of radius r $$m'(r)=\frac{4}{3}\pi r^3 m_s n$$ Thus we have $$v(r)=\left(\frac{4\pi G n m_s}{3}\right)^{1/2} r$$	0.2 pts
While for particle outside the sphere, we have $$v(r)=\left(\frac{4\pi G n m_s R^3}{3r}\right)^{1/2}$$	0.2 pts

The sketch is given below  Sketch of the rotation velocity vs distance from the center of galaxy	0.1 pts
Total	0.8 pts

Question B.2

Answer	Marks
The total mass can be inferred from $$G\frac{m'(R_g)m_s}{R_g^{\;2}}=\frac{m_s v_0^2}{R_g}$$ Thus $$m_R=m'(R_g)=\frac{v_0^2 R_g}{G}$$	0.5 pts
Total	0.5 pts

Question B.3

Answer	Marks
Base on the previous answer in B.1, if the mass of the galaxy comes only from the visible stars, then the galaxy rotation curve should fall proportional to $1/\sqrt{r}$ on the outside at a distance $r > R_g$. But in the figure of problem b) the curve remain constant after $r > R_g$, we can infer from $$G\frac{m'(r)m_s}{r^2} = \frac{m_s v_0^2}{r}.$$ to make $v(r)$ constant, then $m'(r)$ should be proportional to r for $r > R_g$, i.e. for $r > R_g$, $m'(r) = Ar$ with A is a constant.	0.3 pts
While for $r < R_g$, to obtain a linear plot proportional to r, then $m'(r)$ should be proportional to r^3 , i.e. $m'(r) = Br^3$.	0.3 pts
Thus for $r < R_g$ we have $$m'(r) = \int_0^r \rho_t(r) 4\pi r'^2\, dr' = Br^3$$ $$dm'(r) = \rho_t(r) 4\pi r^2 dr = 3Br^2 dr$$ Thus total mass density $\rho_t(r) = \frac{3B}{4\pi}$	0.2 pts
$$m_R = \int_0^{R_g} \frac{3B}{4\pi} 4\pi r'^2\, dr' = BR_g^{\ 3} \text{ or } B = \frac{m_R}{R_g^{\ 3}} = \frac{v_0^2}{GR_g^{\ 2}}$$ Thus the dark matter mass density $\rho(r) = \frac{3v_0^2}{4\pi G R_g^{\ 2}} - nm_s$	0.2 pts

While for $r > R_g$ we have $m'(r) = \int_0^{R_g} \rho(r')4\pi r'^2\, dr' + \int_{R_g}^{r} \rho(r')4\pi r'^2\, dr' = Ar$ $m'(r) = m_R + \int_{R_g}^{r} \rho(r')4\pi r'^2\, dr' = Ar$ $\int_R^r \rho(r')4\pi r'^2\, dr' = Ar - M_0$ $\rho(r)4\pi r^2 = A$, or $\rho(r) = \dfrac{A}{4\pi r^2}$.	0.2 pts
Now to find the constant *A*. $\int_R^r \dfrac{A}{4\pi r'^2} 4\pi r'^2\, dr' = A(r - R_g) = Ar - m_R$ Thus $AR_g = m_R$ and $A = \dfrac{v_0^2}{G}$ We can also find *A* from the following $G\dfrac{m'(r)m_s}{r^2} = G\dfrac{Arm_s}{r^2} = \dfrac{m_s v_0^2}{r}$, thus $A = \dfrac{v_0^2}{G}$. Thus the dark matter mass density (which is also the total mass density since $n \approx 0$ for $r \geq R_g$. $\rho(r) = \dfrac{v_0^2}{4\pi G r^2}$ for $r \geq R_g$	0.3 pts
Total	1.5 pts

C. Interstellar Gas and Dark Matter

Question C.1

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Consider a very small volume of a disk with area A and thickness Δr, see Fig.1

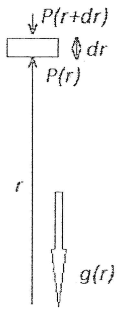

Figure 1. Hydrostatic equilibrium

In hydrostatic equilibrium we have

$$(P(r)-P(r+\Delta r))A-\rho g(r)A\Delta r=0$$

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$$\frac{\Delta P}{\Delta r}=-\rho\frac{Gm'(r)}{r^2}$$

$$\frac{dP}{dr}=-\rho\frac{Gm'(r)}{r^2}=-n(r)m_p\frac{Gm'(r)}{r^2}.$$

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Question C.2

Answer	Marks
Using the ideal gas law $P = n\,kT$ where $n = N/V$ where n is the number density, we have $$\frac{dP}{dr} = kT\frac{dn(r)}{dr} + kn(r)\frac{dT}{dr} = -n(r)m_p\frac{Gm'(r)}{r^2}$$ Thus we have $$m'(r) = -\frac{kT}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr} + \frac{r^2}{T(r)}\frac{dT(r)}{dr}\right).$$	0.5 pts
Total	0.5 pts

Question C.3

Answer	Marks
If we have isothermal distribution, we have $dT/dr = 0$ and $$m'(r) = -\frac{kT_0}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr}\right)$$	0.2 pts
From information about interstellar gas number density, we have $$\frac{1}{n(r)}\frac{dn(r)}{dr} = -\frac{3r+\beta}{r(r+\beta)}$$ Thus we have $$m'(r) = \frac{kT_0 r}{Gm_p}\frac{3r+\beta}{(r+\beta)}$$	0.2 pts

Mass density of the interstellar gas is $\rho_g(r)=\dfrac{\alpha m_p}{r(\beta+r)^2}$ Thus $m'(r)=\int_0^r\left(\rho_g(r')+\rho_{dm}(r')\right)4\pi r'^2\,dr'=\dfrac{kT_0 r}{Gm_p}\dfrac{3r+\beta}{(r+\beta)}$ $m'(r)=\int_0^r\left(\dfrac{\alpha m_p}{r'(\beta+r')^2}+\rho_{dm}(r')\right)4\pi r'^2\,dr'=\dfrac{kT_0 r}{Gm_p}\dfrac{3r+\beta}{(r+\beta)}$	0.3 pts
$\left(\dfrac{\alpha m_p}{r(\beta+r)^2}+\rho_{dm}(r)\right)4\pi r^2=\dfrac{kT_0}{Gm_p}\dfrac{3r^2+6r\beta+\beta^2}{(r+\beta)^2}$ $\rho_{dm}(r)=\dfrac{kT_0}{4\pi Gm_p}\dfrac{3r^2+6r\beta+\beta^2}{(r+\beta)^2 r^2}-\dfrac{\alpha m_p}{r(\beta+r)^2}$	0.3 pts
Total	1.0 pts