Speed of light (solution) E1

In this document decimal comma is used instead of decimal point in graphs and tables

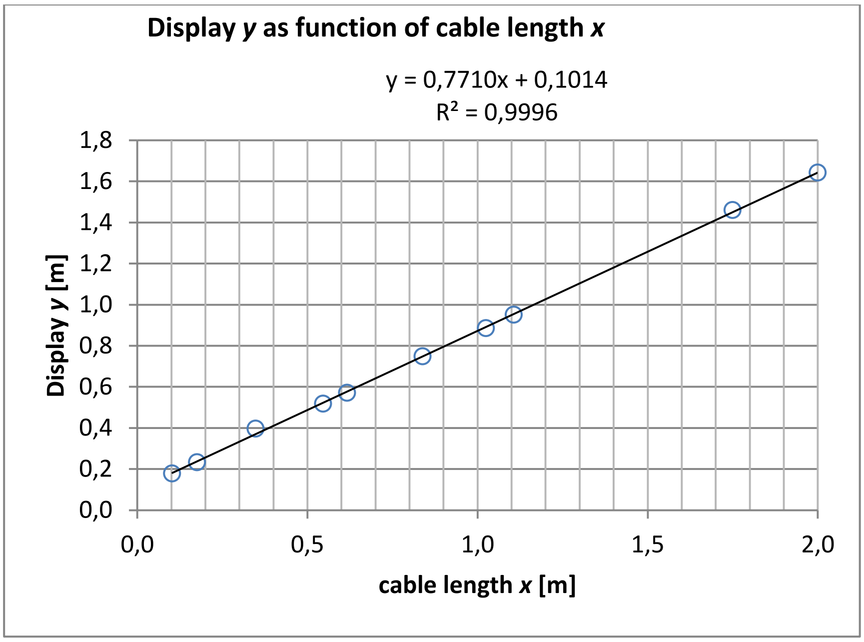
1.1	Use the LDM to measure the distance H from the top of the table to the floor. Write down the uncertainty ΔH. Show with a sketch how you perform this measurement.	0.4

$H = 907\text{ mm} \pm 2\text{ mm}$. See the sketch in the figure corresponding to 1.3b. It must appear how the height is measured with the LDM in the rear mode.

1.2a	Measure corresponding values of x and y. Set up a table with your measurements. Draw a graph showing y as a function of x.	1.8

Here, a 2 m cable is used, but 1 m is sufficient. There should be about 8 lengths evenly distributed in the interval from 0 m to 1 m.

x	y
m	m
0,103	0,177
0,176	0,232
0,348	0,396
0,546	0,517
0,617	0,570
0,839	0,748
1,025	0,885
1,107	0,950
1,750	1,459
2,000	1,642



1.2b	Use the graph to find the refractive index n_{co} for the material from which the core of the fiber optic cable is made. Calculate the speed of light v_{co} in the core of the fiber optic cable.	1.2

The refractive index is twice the gradient of the linear graph, $n_{co} = 2 \cdot 0.7710 \approx 1.54$.

The reason for that is that the travel time for a light pulse

$$t = \frac{x}{v_{co}} = \frac{x n_{co}}{c}$$

The display will therefore show $y = \frac{1}{2} ct + k \Leftrightarrow y = \frac{1}{2} n_{co} x + k$.

The speed of light in the core of the cable is $v_{co} = \frac{c}{n_{co}} = \frac{2{,}998 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}}}{2 \cdot 0.7710} \approx 1.95 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}}$

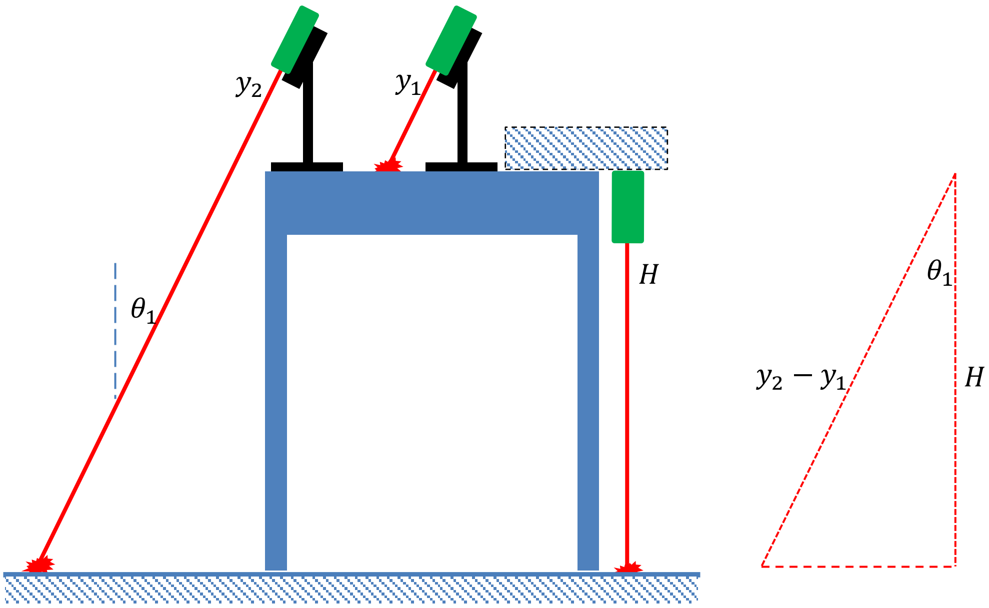
1.3a	Measure with the LDM the distance y_1 to the laser dot where the laser beam hits the table top. Then move the box with the LDM horizontally until the laser beam hits the floor. Measure the distance y_2 to the laser dot where the laser beam hits the floor. State the uncertainties.	0.2

$y_1 = 312\ \mathrm{mm} \pm 2\ \mathrm{mm}$, $y_2 = 1273\ \mathrm{mm} \pm 2\ \mathrm{mm}$

1.3b	Calculate the angle θ_1 using only these measurements y_1, y_2 and H (from problem 1.1). Determine the uncertainty $\Delta\theta_1$.	0.4

$$\theta_1 = \cos^{-1}\left(\frac{H}{y_2 - y_1}\right) = \cos^{-1}\left(\frac{907\ \mathrm{mm}}{961\ \mathrm{mm}}\right) = 19.30°$$

(see the figure)



Measuring the horizontal part of some triangle is very inaccurate because of the size of the laser dot. No marks will be awarded for that. Using $\delta = 2$ mm as the uncertainty of y_1, y_2 and H, the uncertainty of θ_1 can be calculated as follows:

$$\Delta \cos\theta_1 = \Delta\left(\frac{H}{y_2 - y_1}\right)$$

Using simple derivatives yields

$$\tan\theta_1 \cdot \Delta\theta_1 = \frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}$$

$$\Delta\theta_1 = \frac{\left(\frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}\right)}{\tan\theta_1} \cdot \frac{180°}{\pi} = \frac{\left(\frac{2}{907} + \frac{4}{961}\right)}{\tan 19{,}30°} \cdot \frac{180°}{\pi} \approx 1°$$

Otherwise, using min/max method

$$\Delta\theta_1 = \theta_{1\max} - \theta_1 = \cos^{-1}\left(\frac{H_{\min}}{y_{2\max} - y_{1\min}}\right) = \cos^{-1}\left(\frac{905\text{ mm}}{965\text{ mm}}\right) - \cos^{-1}\left(\frac{907\text{ mm}}{961\text{ mm}}\right) = 1.0°$$

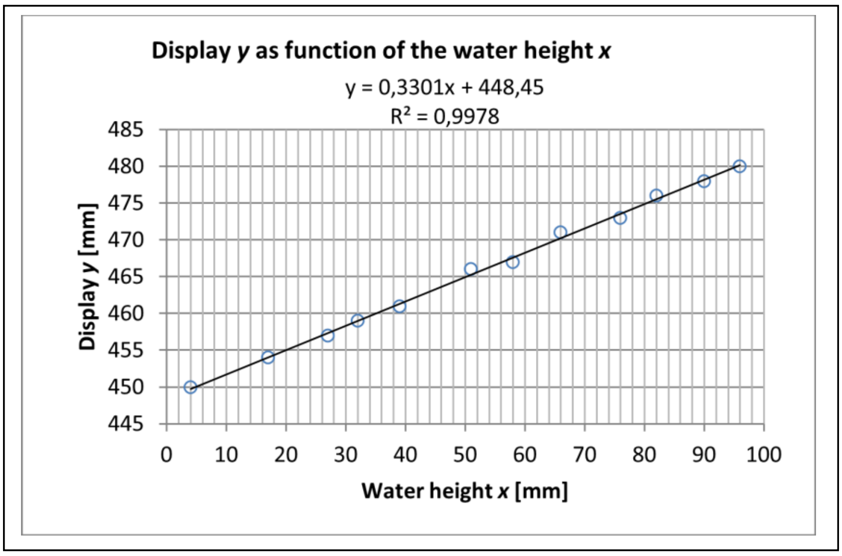
Alternatively, calculate $\Delta\theta_1$ using $\Delta(y_2 - y_1) = \sqrt{(\Delta y_1)^2 + (\Delta y_2)^2} = \sqrt{2}\delta$ and then

$$\tan\theta_1 \cdot \Delta\theta_1 = \sqrt{\left(\frac{\delta}{H}\right)^2 + \frac{2\delta^2}{(y_2 - y_1)^2}}$$

Also, accept $\delta = 1$ mm and $\Delta\theta_1 = 0.5°$.

1.4a	Measure corresponding values of x and y. Set up a table with your measurements. Draw a graph of y as a function of x.	1.6

x [mm]	y [mm]
4	450
17	454
27	457
32	459
39	461
51	466
58	467
66	471
76	473
82	476
90	478
96	480



1.4b	Use equations to explain theoretically what the graph is expected to look like.	1.2

The time it takes the light to reach the water surface is

$$t_1 = \frac{(h-x)/\cos\theta_1}{c}$$

From the water surface to the bottom the light uses the time

$$t_2 = \frac{x/\cos\theta_2}{v}$$

Total travel time forth and back

$$t = 2t_1 + 2t_2 = 2\frac{(h-x)/\cos\theta_1}{c} + 2\frac{x/\cos\theta_2}{v} = 2\frac{h-x}{c\cos\theta_1} + 2\frac{nx}{c\cos\theta_2}$$

Hence, the display will show (we simply write $n = n_{\mathrm{w}}$)

$$y = ½ct + k = \left(\frac{n}{\cos\theta_2} - \frac{1}{\cos\theta_1}\right)x + \frac{h}{\cos\theta_1} + k$$

which is a linear function of x. Then, using a trigonometric identity and Snell's law,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{\sin^2\theta_1}{n^2}}.$$

From this the gradient α is found to be

$$\alpha = \frac{n}{\sqrt{1-\frac{\sin^2\theta_1}{n^2}}} - \frac{1}{\cos\theta_1} = \frac{n^2}{\sqrt{n^2-\sin^2\theta_1}} - \frac{1}{\cos\theta_1}$$

1.4c	Use the graph to determine the refractive index n_{w} for water.	1.2

Knowing the gradient α from the graph, the index of refraction n is found by solving this equation. Introducing a practical parameter,

$$p = \alpha + \frac{1}{\cos\theta_1}$$

the above equation becomes

$$p = \frac{n_{\mathrm{w}}^2}{\sqrt{n_{\mathrm{w}}^2 - \sin^2\theta_1}} \Leftrightarrow n_{\mathrm{w}}^4 - p^2 n_{\mathrm{w}}^2 + p^2\sin^2\theta_1 = 0$$

with the solution

$$n_{\mathrm{w}} = \sqrt{\frac{p^2 \pm \sqrt{p^4 - 4p^2\sin^2\theta_1}}{2}} = \frac{\sqrt{2}}{2}p\sqrt{1 \pm \sqrt{1-\left(\frac{2\sin\theta_1}{p}\right)^2}}$$

From the graph is found $\alpha = 0.3301$, which leads to $p = 1.388356$ and hence

$$n_{\mathrm{w}} = 1.34676 \approx 1.347.$$

All solutions with $n_{\mathrm{w}} < 1$ are omitted.

Another and more elegant way of finding n_w is to use Snell's law in the equation

$$\alpha = \frac{n_\mathrm{w}}{\cos\theta_2} - \frac{1}{\cos\theta_1} = \frac{\sin\theta_1}{\sin\theta_2\cos\theta_2} - \frac{1}{\cos\theta_1} = \frac{2\sin\theta_1}{\sin 2\theta_2} - \frac{1}{\cos\theta_1}$$

This yields

$$\sin 2\theta_2 = \frac{2\sin\theta_1}{\alpha + \dfrac{1}{\cos\theta_1}}$$

From here θ_2 can be calculated leading to $n_\mathrm{w} = \frac{\sin\theta_1}{\sin\theta_2}$. This method also only uses the graph and the angle θ_1, and measurement of θ_2 is not involved).

The table value for pure water at normal conditions is $n_\mathrm{w} = 1.331$ at the wavelength $\lambda = 635\ \mathrm{nm}$.

The following approximations can be used: For small angles

$$n_\mathrm{w} \approx \frac{\sqrt{2}}{2}p\sqrt{1 + 1 - \frac{1}{2}\left(\frac{2\sin\theta_1}{p}\right)^2} \approx p\sqrt{1 - \left(\frac{\sin\theta_1}{p}\right)^2} \approx p\left(1 - \frac{1}{2}\left(\frac{\sin\theta_1}{p}\right)^2\right)$$

For very small angles, we get

$$n_\mathrm{w} \approx p \approx \alpha + 1$$

It is much simpler, but not recommendable, to do the experiment with very small $\theta_1 \approx 0$. Reflections in the water surface will ruin the signal from the bottom.