Solution to Theoretical Question 1

A Swing with a Falling Weight

## *Part A*

(a) Since the length of the string $L = s + R\theta$ is constant, its rate of change must be zero. Hence we have

$$\dot{s} + R\dot{\theta} = 0 \tag{A1}$$

(b) Relative to *O*, *Q* moves on a circle of radius *R* with angular velocity $\dot{\theta}$, so

$$\vec{v}_Q = R\dot{\theta}\hat{t} = -\dot{s}\hat{t} \tag{A2}$$

(c) Refer to Fig. A1. Relative to *Q*, the displacement of *P* in a time interval $\Delta t$ is $\Delta\vec{r}\,' = (s\Delta\theta)(-\hat{r}) + (\Delta s)\hat{t} = [(s\dot{\theta})(-\hat{r}) + \dot{s}\hat{t}]\Delta t$. It follows

$$\vec{v}\,' = -s\dot{\theta}\hat{r} + \dot{s}\hat{t} \tag{A3}$$

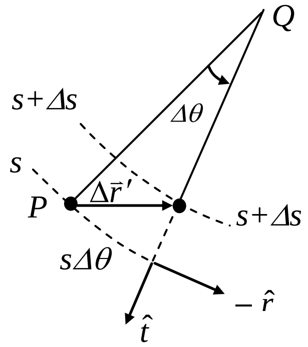



Figure A1

(d) The velocity of the particle relative to *O* is the sum of the two relative velocities given in Eqs. (A2) and (A3) so that

$$\vec{v} = \vec{v}\,' + \vec{v}_Q = (-s\dot{\theta}\hat{r} + \dot{s}\hat{t}) + R\dot{\theta}\hat{t} = -s\dot{\theta}\hat{r} \tag{A4}$$

(e) Refer to Fig. A2. The ( $-\hat{t}$ )-component of the velocity change $\Delta\vec{v}$ is given by $(-\hat{t})\cdot\Delta\vec{v} = v\Delta\theta = v\dot{\theta}\Delta t$. Therefore, the $\hat{t}$-component of the acceleration $\vec{a} = \Delta\vec{v}/\Delta t$ is given by $\hat{t}\cdot\vec{a} = -v\dot{\theta}$. Since the speed *v* of the particle is $s\dot{\theta}$ according to Eq. (A4), we see that the $\hat{t}$-component of the particle's acceleration at *P* is given by

$$\vec{a}\cdot\hat{t} = -v\dot{\theta} = -(s\dot{\theta})\dot{\theta} = -s\dot{\theta}^2 \tag{A5}$$

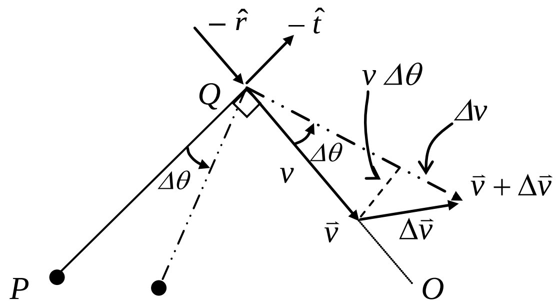



Figure A2

Note that, from Fig. A2, the radial component of the acceleration may also be obtained as

$\bar{a} \cdot \hat{r} = -dv/dt = -d(s\dot{\theta})/dt$.

(f) Refer to Fig. A3. The gravitational potential energy of the particle is given by $U = -mgh$. It may be expressed in terms of $s$ and $\theta$ as

$$U(\theta) = -mg[R(1-\cos\theta) + s\sin\theta] \tag{A6}$$

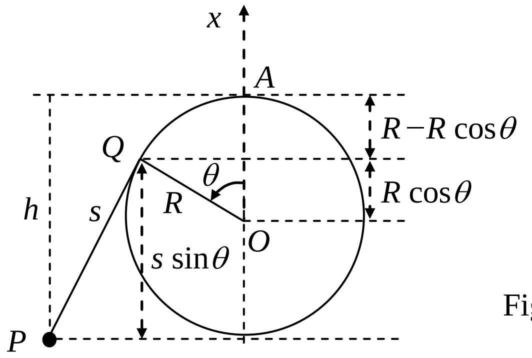



Figure A3

(g) At the lowest point of its trajectory, the particle's gravitational potential energy $U$ must assume its minimum value $U_m$. By differentiating Eq. (A6) with respect to $\theta$ and using Eq. (A1), the angle $\theta_m$ corresponding to the minimum gravitational energy can be obtained.

$$\begin{aligned}\frac{dU}{d\theta} &= -mg\left(R\sin\theta + \frac{ds}{d\theta}\sin\theta + s\cos\theta\right) \\ &= -mg\left[R\sin\theta + (-R)\sin\theta + s\cos\theta\right] \\ &= -mgs\cos\theta\end{aligned}$$

At $\theta = \theta_m$, $\left.\frac{dU}{d\theta}\right|_{\theta_m} = 0$. We have $\theta_m = \frac{\pi}{2}$. The lowest point of the particle's trajectory is shown in Fig. A4 where the length of the string segment of QP is $s = L - \pi R/2$.

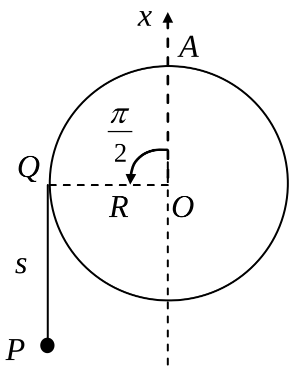



Figure A4

From Fig. A4 or Eq. (A6), the minimum potential energy is then

$$U_m = U(\pi/2) = -mg[R + L - (\pi R/2)] \tag{A7}$$

Initially, the total mechanical energy $E$ is 0. Since $E$ is conserved, the speed $v_m$ of the particle at the lowest point of its trajectory must satisfy

$$E = 0 = \frac{1}{2} m v_m^2 + U_m \tag{A8}$$

From Eqs. (A7) and (A8), we obtain

$$v_m = \sqrt{-2U_m / m} = \sqrt{2g[R + (L - \pi R/2)]} \tag{A9}$$

## *Part B*

(h) From Eq. (A6), the total mechanical energy of the particle may be written as

$$E = 0 = \frac{1}{2} m v^2 + U(\theta) = \frac{1}{2} m v^2 - mg[R(1-\cos\theta) + s \sin\theta] \tag{B1}$$

From Eq. (A4), the speed $v$ is equal to $s\dot{\theta}$. Therefore, Eq. (B1) implies

$$v^2 = (s\dot{\theta})^2 = 2g[R(1-\cos\theta) + s \sin\theta] \tag{B2}$$

Let $T$ be the tension in the string. Then, as Fig. B1 shows, the $\hat{t}$-component of the net force on the particle is $-T + mg \sin\theta$. From Eq. (A5), the tangential acceleration of the particle is $(-s\dot{\theta}^2)$. Thus, by Newton's second law, we have

$$m(-s\dot{\theta}^2) = -T + mg \sin\theta \tag{B3}$$

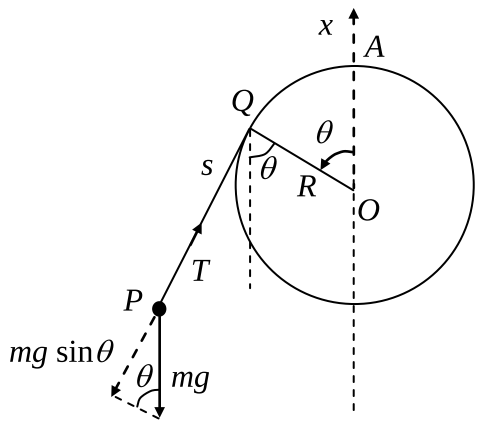
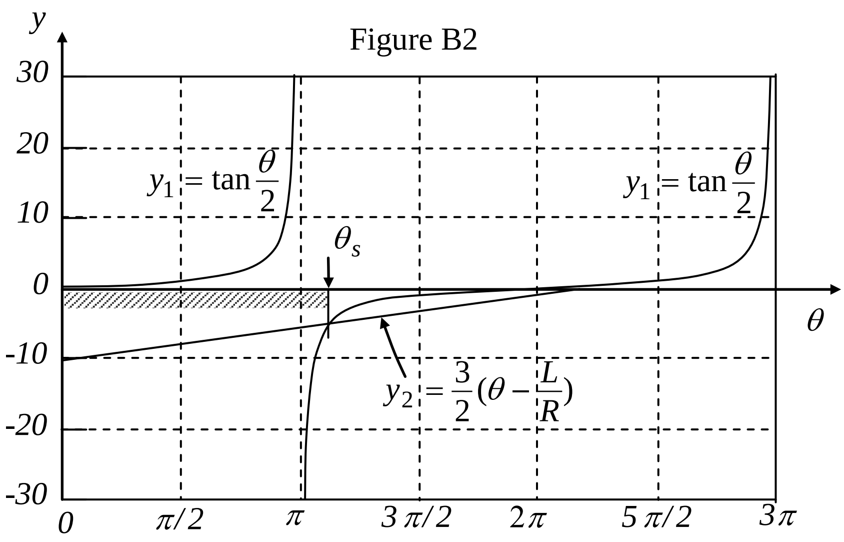



Figure B1

According to the last two equations, the tension may be expressed as

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g \sin\theta) = \frac{mg}{s}[2R(1-\cos\theta) + 3s\sin\theta] \\ &= \frac{2mgR}{s}\left[\tan\frac{\theta}{2} - \frac{3}{2}\left(\theta - \frac{L}{R}\right)\right](\sin\theta) \\ &= \frac{2mgR}{s}(y_1 - y_2)(\sin\theta) \end{aligned} \tag{B4}$$

The functions $y_1 = \tan(\theta/2)$ and $y_2 = 3(\theta - L/R)/2$ are plotted in Fig B2.




From Eq. (B4) and Fig. B2, we obtain the result shown in Table B1. The angle at which .$y_2 = y_1$ is called $\theta_s$ ( $\pi < \theta_s < 2\pi$ ) and is given by

$$\frac{3}{2}(\theta_s - \frac{L}{R}) = \tan\frac{\theta_s}{2} \tag{B5}$$

or, equivalently, by

$$\frac{L}{R} = \theta_s - \frac{2}{3}\tan\frac{\theta_s}{2} \tag{B6}$$

Since the ratio $L/R$ is known to be given by

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} = (\pi + \frac{\pi}{8}) - \frac{2}{3}\tan\frac{1}{2}(\pi + \frac{\pi}{8}) \tag{B7}$$

one can readily see from the last two equations that $\theta_s = 9\pi/8$.

Table B1

| | $(y_1 - y_2)$ | $\sin\theta$ | tension $T$ |
|---|---|---|---|
| $0 < \theta < \pi$ | positive | positive | positive |
| $\theta = \pi$ | $+\infty$ | 0 | positive |
| $\pi < \theta < \theta_s$ | negative | negative | positive |
| $\theta = \theta_s$ | zero | negative | zero |
| $\theta_s < \theta < 2\pi$ | positive | negative | negative |

Table B1 shows that the tension $T$ must be positive (or the string must be taut and straight) in the angular range $0 < \theta < \theta_s$. Once $\theta$ reaches $\theta_s$, the tension $T$ becomes zero and the part of the string not in contact with the rod will not be straight afterwards. The shortest possible value $s_{\min}$ for the length $s$ of the line segment $QP$ therefore occurs at $\theta = \theta_s$ and is given by

$$s_{\min} = L - R\theta_s = R(\frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} - \frac{9\pi}{8}) = \frac{2R}{3}\cot\frac{\pi}{16} = 3.352R \qquad \text{(B8)}$$

When $\theta = \theta_s$, we have $T = 0$ and Eqs. (B2) and (B3) then leads to $v_s^2 = -gs_{\min}\sin\theta_s$.
Hence the speed $v_s$ is

$$v_s = \sqrt{-gs_{\min}\sin\theta_s} = \sqrt{\frac{2gR}{3}\cot\frac{\pi}{16}\sin\frac{\pi}{8}} = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16} \qquad \text{(B9)}$$
$$= 1.133\sqrt{gR}$$

(i) When $\theta \ge \theta_s$, the particle moves like a projectile under gravity. As shown in Fig. B3, it is projected with an initial speed $v_s$ from the position $P = (x_s, y_s)$ in a direction making an angle $\phi = (3\pi/2 - \theta_s)$ with the $y$-axis.

The speed $v_H$ of the particle at the highest point of its parabolic trajectory is equal to the $y$-component of its initial velocity when projected. Thus,

$$v_H = v_s\sin(\theta_s - \pi) = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16}\sin\frac{\pi}{8} = 0.4334\sqrt{gR} \qquad \text{(B10)}$$

The horizontal distance $H$ traveled by the particle from point $P$ to the point of maximum height is

$$H = \frac{v_s^2\sin 2(\theta_s - \pi)}{2g} = \frac{v_s^2}{2g}\sin\frac{9\pi}{4} = 0.4535R \qquad \text{(B11)}$$

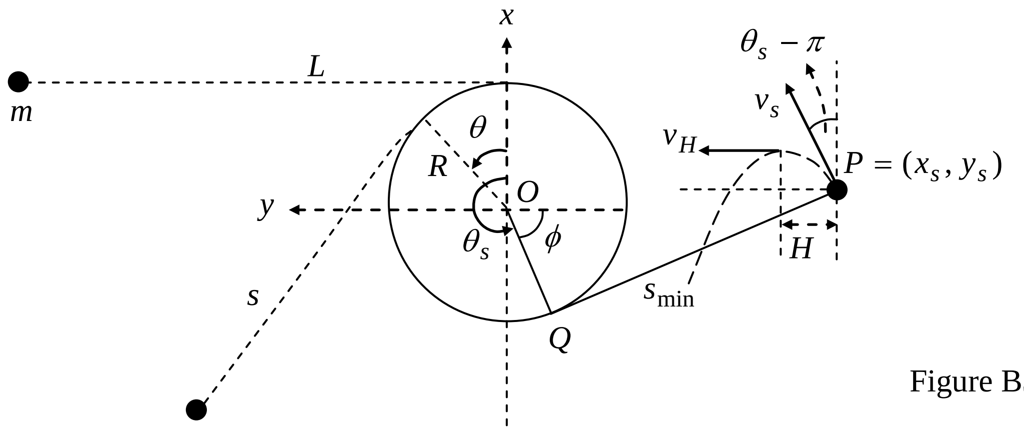



Figure B3

The coordinates of the particle when $\theta = \theta_s$ are given by

$$x_s = R\cos\theta_s - s_{\min}\sin\theta_s = -R\cos\frac{\pi}{8} + s_{\min}\sin\frac{\pi}{8} = 0.358R \qquad \text{(B12)}$$

$$y_s = R\sin\theta_s + s_{\min}\cos\theta_s = -R\sin\frac{\pi}{8} - s_{\min}\cos\frac{\pi}{8} = -3.478R \qquad \text{(B13)}$$

Evidently, we have $|y_s| > (R + H)$. Therefore the particle can indeed reach its maximum height without striking the surface of the rod.

## *Part C*

(j) Assume the weight is initially lower than $O$ by $h$ as shown in Fig. C1.

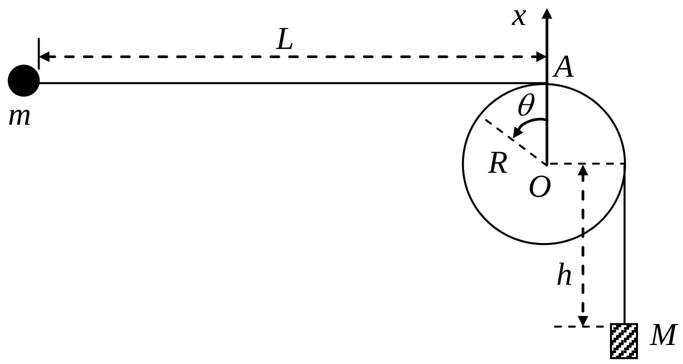



Figure C1

When the weight has fallen a distance $D$ and stopped, the law of conservation of total mechanical energy as applied to the particle-weight pair as a system leads to

$$-Mgh = E' - Mg(h+D) \qquad \text{(C1)}$$

where $E'$ is the *total mechanical energy of the particle* when the weight has stopped. It follows

$$E' = MgD \qquad \text{(C2)}$$

Let $\Lambda$ be the total length of the string. Then, its value at $\theta = 0$ must be the same as at any other angular displacement $\theta$. Thus we must have

$$\Lambda = L + \frac{\pi}{2}R + h = s + R(\theta + \frac{\pi}{2}) + (h+D) \qquad \text{(C3)}$$

Noting that $D = \alpha L$ and introducing $\ell = L - D$, we may write

$$\ell = L - D = (1-\alpha)L \qquad \text{(C4)}$$

From the last two equations, we obtain

$$s = L - D - R\theta = \ell - R\theta \qquad \text{(C5)}$$

After the weight has stopped, the total mechanical energy of the particle must be conserved. According to Eq. (C2), we now have, instead of Eq. (B1), the following equation:

$$E' = MgD = \frac{1}{2}mv^2 - mg\left[R(1-\cos\theta) + s\sin\theta\right] \qquad \text{(C6)}$$

The square of the particle's speed is accordingly given by

$$v^2 = (s\dot{\theta})^2 = \frac{2MgD}{m} + 2gR\left[(1-\cos\theta) + \frac{s}{R}\sin\theta\right] \qquad \text{(C7)}$$

Since Eq. (B3) stills applies, the tension $T$ of the string is given by

$$-T + mg\sin\theta = m(-s\dot{\theta}^2) \qquad \text{(C8)}$$

From the last two equations, it follows

$$
\begin{aligned}
T &= m(s\dot{\theta}^{2} + g\sin\theta) \\
&= \frac{mg}{s}\left[\frac{2M}{m}D + 2R(1-\cos\theta) + 3s\sin\theta\right] \\
&= \frac{2mgR}{s}\left[\frac{MD}{mR} + (1-\cos\theta) + \frac{3}{2}\left(\frac{\ell}{R}-\theta\right)\sin\theta\right]
\end{aligned} \tag{C9}
$$

where Eq. (C5) has been used to obtain the last equality.

We now introduce the function

$$
f(\theta) = 1 - \cos\theta + \frac{3}{2}\left(\frac{\ell}{R}-\theta\right)\sin\theta \tag{C10}
$$

From the fact $\ell = (L-D) >> R$, we may write

$$
f(\theta) \approx 1 + \frac{3}{2}\frac{\ell}{R}\sin\theta - \cos\theta = 1 + A\sin(\theta-\phi) \tag{C11}
$$

where we have introduced

$$
A = \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^{2}} \quad , \quad \phi = \tan^{-1}\left(\frac{2R}{3\ell}\right) \tag{C12}
$$

From Eq. (C11), the minimum value of $f(\theta)$ is seen to be given by

$$
f_{\min} = 1 - A = 1 - \sqrt{1 + \left(\frac{3}{2}\frac{\ell}{R}\right)^{2}} \tag{C13}
$$

Since the tension $T$ remains nonnegative as the particle swings around the rod, we have from Eq. (C9) the inequality

$$
\frac{MD}{mR} + f_{\min} = \frac{M(L-\ell)}{mR} + 1 - \sqrt{1 + \left(\frac{3\ell}{2R}\right)^{2}} \geq 0 \tag{C14}
$$

or

$$
\left(\frac{ML}{mR}\right) + 1 \;\geq\; \left(\frac{M\ell}{mR}\right) + \sqrt{1 + \left(\frac{3\ell}{2R}\right)^{2}} \approx \left(\frac{M\ell}{mR}\right) + \left(\frac{3\ell}{2R}\right) \tag{C15}
$$

From Eq. (C4), Eq. (C15) may be written as

$$
\left(\frac{ML}{mR}\right) + 1 \;\geq\; \left(\frac{ML}{mR} + \frac{3L}{2R}\right)(1-\alpha) \tag{C16}
$$

Neglecting terms of the order ($R/L$) or higher, the last inequality leads to

$$
\alpha \;\geq\; 1 - \frac{\left(\frac{ML}{mR}\right)+1}{\left(\frac{ML}{mR}+\frac{3L}{2R}\right)} = \frac{\frac{3L}{2R}-1}{\frac{ML}{mR}+\frac{3L}{2R}} = \frac{1-\frac{2R}{3L}}{\frac{2M}{3m}+1} \approx \frac{1}{1+\frac{2M}{3m}} \tag{C17}
$$

The critical value for the ratio $D/L$ is therefore

$$
\alpha_{c} = \frac{1}{1+\frac{2M}{3m}} \tag{C18}
$$

## Marking Scheme

### Theoretical Question 1
### A Swing with a Falling Weight

| Total Scores | Sub Scores | Marking Scheme for Answers to the Problem |
|---|---|---|
| Part A<br><br>4.3 pts. | (a)<br><br>0.5 | Relation between $\dot{\theta}$ and $\dot{s}$. ($\dot{s} = -R\dot{\theta}$)<br>➢ 0.2 for $\dot{\theta} \propto \dot{s}$.<br>➢ 0.3 for proportionality constant (-*R*). |
| | (b)<br><br>0.5 | Velocity of *Q* relative to *O*. ($\vec{v}_Q = R\dot{\theta}\hat{t}$)<br>➢ 0.2 for magnitude $R\dot{\theta}$.<br>➢ 0.3 for direction $\hat{t}$. |
| | (c)<br><br>0.7 | Particle's velocity at *P* relative to *Q*. ($\vec{v}' = -s\dot{\theta}\hat{r} + \dot{s}\hat{t}$)<br>➢ 0.2+0.1 for magnitude and direction of $\hat{r}$-component.<br>➢ 0.3+0.1 for magnitude and direction of $\hat{t}$-component. |
| | (d)<br>0.7 | Particle's velocity at *P* relative to *O*. ($\vec{v} = \vec{v}' + \vec{v}_Q = -s\dot{\theta}\hat{r}$)<br>➢ 0.3 for vector addition of $\vec{v}'$ and $\vec{v}_Q$.<br>➢ 0.2+0.2 for magnitude and direction of $\vec{v}$. |
| | (e)<br><br>0.7 | $\hat{t}$-component of particle's acceleration at *P*.<br>➢ 0.3 for relating $\vec{a}$ or $\vec{a} \cdot \hat{t}$ to the velocity in a way that implies $\lvert\vec{a} \cdot \hat{t}\rvert = v^2 / s$.<br>➢ 0.4 for $\vec{a} \cdot \hat{t} = -s\dot{\theta}^2$ (0.1 for minus sign.) |
| | (f)<br><br>0.5 | Potential energy *U*.<br>➢ 0.2 for formula $U = -mgh$.<br>➢ 0.3 for $h = R(1-\cos\theta) + s\sin\theta$ or *U* as a function of $\theta$, *s*, and *R*. |
| | (g)<br><br>0.7 | Speed at lowest point $v_m$.<br>➢ 0.2 for lowest point at $\theta = \pi/2$ or *U* equals minimum $U_m$.<br>➢ 0.2 for total mechanical energy $E = mv_m^2/2 + U_m = 0$.<br>➢ 0.3 for $v_m = \sqrt{-2U_m/m} = \sqrt{2g[R+(L-\pi R/2)]}$. |
| Part B<br><br>4.3 pts. | (h)<br><br>2.4 | Particle's speed $v_s$ when $\overline{QP}$ is shortest.<br>➢ 0.4 for tension *T* becomes zero when $\overline{QP}$ is shortest.<br>➢ 0.3 for equation of motion $-T + mg\sin\theta = m(-s\dot{\theta}^2)$.<br>➢ 0.3 for $E = 0 = m(s\dot{\theta})^2/2 - mg[R(1-\cos\theta) + s\sin\theta]$.<br>➢ 0.4 for $\frac{3}{2}(\theta_s - \frac{L}{R}) = \tan\frac{\theta_s}{2}$.<br>➢ 0.5 for $\theta_s = 9\pi/8$.<br>➢ 0.3+0.2 for $v_s = \sqrt{4gR/3}\cos\pi/16 = 1.133\sqrt{gR}$ |

| | | |
|---|---|---|
| | (i)<br><br>1.9 | The speed $v_H$ of the particle at its highest point.<br>➢ 0.4 for particle undergoes projectile motion when $\theta \geq \theta_s$.<br>➢ 0.3 for angle of projection $\phi = (3\pi/2 - \theta_s)$.<br>➢ 0.3 for $v_H$ is the *y*-component of its velocity at $\theta = \theta_s$.<br>➢ 0.4 for noting particle does not strike the surface of the rod.<br>➢ 0.3+0.2 for<br>$v_H = \sqrt{4gR/3}\cos(\pi/16)\sin(\pi/8) = 0.4334\sqrt{gR}$. |
| Part C<br><br>3.4 pts | (j)<br><br>3.4 | The critical value $\alpha_c$ of the ratio $D/L$.<br>➢ 0.4 for particle's energy $E' = MgD$ when the weight has stopped.<br>➢ 0.3 for $s = L - D - R\theta$.<br>➢ 0.3 for $E' = MgD = mv^2/2 - mg[R(1-\cos\theta) + s\sin\theta]$.<br>➢ 0.3 for $-T + mg\sin\theta = m(-s\dot{\theta}^2)$.<br>➢ 0.3 for concluding $T$ must not be negative.<br>➢ 0.6 for an inequality leading to the determination of the range of $D/L$.<br>➢ 0.6 for solving the inequality to give the range of $\alpha = D/L$.<br>➢ 0.6 for $\alpha_c = (1 + 2M/3m)$. |