Question 1

A *Bungee Jumper*

(a) The jumper comes to rest when

lost gravitational potential energy = stored strain energy

$$mgy = \tfrac{1}{2}k(y-L)^2$$

0.1

$$ky^2 - 2y(kL + mg) + kL^2 = 0$$

0.1

This is solved as a quadratic.

$$y = \frac{2(kL + mg) \pm \sqrt{4(kL + mg)^2 - 4k^2L^2}}{2k}$$

$$= \frac{kL + mg \pm \sqrt{2mgkL + m^2g^2}}{k}$$

0.2

Need positive root; lower position of rest (other root after initial rise).

0.1

0.5

(b) The maximum speed is attained when the acceleration is zero and forces balance; i.e. when $mg = kx$ 0.1

Also kinetic energy = lost potential energy – strain energy within elastic rope

0.1

$$\tfrac{1}{2}mv^2 = mg(L + x) - \tfrac{1}{2}kx^2$$

0.1

$$x = \frac{mg}{k}$$

$$v^2 = 2g(L + \frac{mg}{k}) - \frac{mg^2}{k}$$

$$v = \sqrt{2gL + \frac{mg^2}{k}}$$

0.2

0.5

(c) Time to come to rest = time in free fall + time in SHM of rope to stop stretching

0.1

Length of free fall = $L = \tfrac{1}{2}gt_f^{\,2}$

Therefore $t_f = \sqrt{\dfrac{2L}{g}}$

0.2

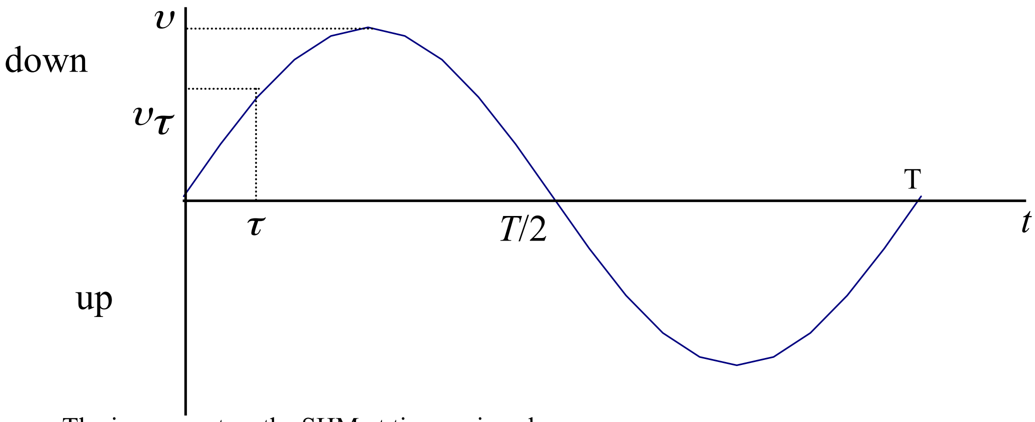
The jumper enters the SHM with free fall velocity $= gt_f = \sqrt{2gL} = v_\tau$

0.1

Period of SHM = $2\pi\sqrt{\frac{m}{k}} = T$

0.1

We represent a full SHM cycle by



The jumper enters the SHM at time τ given by

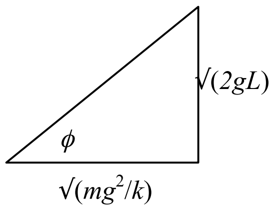
$$\tau = \frac{1}{\omega}\sin^{-1}\frac{\upsilon_\tau}{\upsilon} = \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

0.2

Jumper comes to rest at one half cycle of the SHM at total time given by

$$= t_f + (T/2 \ - \tau)$$

0.1



$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\ \{\pi - \sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\}$$

This is the same as

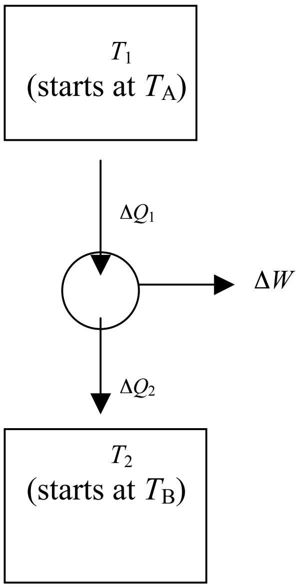
$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\ \{\frac{\pi}{2} + \cos^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\ \tan^{-1}\{-\sqrt{\frac{2kL}{mg}}\}$$

0.2

1.0

B Heat Engine Question



In calculating work obtainable, we assume no loss (friction etc.) in engine working.

ΔQ_1 = energy from body A
$= -ms\Delta T_1$ (ΔT_1 –ve)

$\Delta Q_2 = ms\Delta T_2$ (ΔT_2 +ve)

(a) For maximum amount of mechanical energy assume Carnot engine

$$\frac{\Delta Q_1}{T_1} = \frac{\Delta Q_2}{T_2} \text{ throughout operation (second law)}$$

0.2

But $\Delta Q_1 = -ms\Delta T_1$ and $\Delta Q_2 = ms\Delta T_2$

0.2

$$-ms\int_{T_A}^{T_0}\frac{dT_1}{T_1} = ms\int_{T_B}^{T_0}\frac{dT_2}{T_2}$$

0.1

$$\ln\frac{T_A}{T_0} = \ln\frac{T_0}{T_B}$$

0.1

$$T_0^2 = T_A T_B$$
$$T_0 = \sqrt{T_A T_B}$$

0.2

0.8

$$Q_1 = -ms \int_{T_A}^{T_0} \mathrm{d}T_1 = ms(T_A - T_0)$$

0.2

$$Q_2 = ms \int_{T_B}^{T_0} \mathrm{d}T_2 = ms(T_0 - T_B)$$

0.1

$$W = Q_1 - Q_2$$

0.2

$$W = ms(T_A - T_0 - T_0 + T_B) = ms(T_A + T_B - 2T_0) = ms\,(T_A + T_B - 2\sqrt{T_A T_B}\,)$$

or $ms(\sqrt{T_A} - \sqrt{T_B})^2$

0.2

0.7

(d) Numerical example:

Mass = volume × density

$$W = 2.50 \times 1.00 \times 10^3 \times 4.19 \times 10^3 \times (350 + 300 - 2\sqrt{350 \times 300}\,)\ \text{J}$$

$= 20 \times 10^6$ J

$=$ 20 MJ

0.5

0.5

C Radioactivity and age of the Earth

(a) $N = N_0 e^{-\lambda}$ N_0 = original number

0.1

$n = N_0(1 - e^{-\lambda t})$

0.1

Therefore $n = N e^{\lambda t}(1 - e^{-\lambda t}) = N(e^{\lambda t} - 1)$

0.1

So $n = N(2^{t/\tau} - 1)$ where τ is half-life

or as $\lambda = \dfrac{\ln 2}{T} = \dfrac{0.6931}{T}$, n = N($e^{\frac{0.6931t}{T}} - 1$)

0.1

${}^{206}n = {}^{238}N(2^{t/4.50} - 1)$ or ${}^{206}n = {}^{238}N(e^{0.1540t} - 1)$ where time t is in 10^9 years

0.1

0.5

(b) ${}^{207}n = {}^{235}N(2^{t/0.710} - 1)$ or ${}^{207}n = {}^{235}N(e^{0.9762t} - 1)$

0.1

0.1

(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)

204 : 206 : 207 have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactively produced lead by subtraction

206 : 207 11.7 : 7.1

0.2

Dividing equations from (a) and (b) gives

$$\frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\{\frac{2^{t/4.50} - 1}{2^{t/0.710} - 1}\} \text{ or } \frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\{\frac{e^{0.1540t} - 1}{e^{0.9762t} - 1}\}$$

0.1

$$\frac{11.7}{7.1} = 137\{\frac{2^{T/4.50} - 1}{2^{T/0.710} - 1}\} \text{ or } \frac{11.7}{7.1} = 137\{\frac{e^{0.1540T} - 1}{e^{0.9762T} - 1}\}$$

0.1

$$0.0120\ \{2^{T/0.710} - 1\} = \{2^{T/4.50} - 1\}$$

$$\text{or } 0.0120\ \{e^{0.9762T} - 1\} = \{e^{0.1540T} - 1\}$$

0.1

0.5

(d) Assume $T >> 4.50 \times 10^9$ and ignore 1 in both brackets: 0.2

$$0.0120\ \{2^{T/0.710}\} = \{2^{T/4.50}\} \text{ or } 0.0120\ \{e^{0.9762T}\} = \{e^{0.1540T}\}$$

$$0.0120 = \{2^{T/4.50\ -\ T/0.710}\} = 2^{T(0.222-1.4084)} = 2^{-1.1862T}$$

$$T = -\frac{\log 0.0120}{\log 2 \times 1.1862} = 5.38$$

$T = 5.38 \times 10^9$ years

or $0.0120 = e^{-0.8222T}$ $\quad T = \frac{\ln 0.0120}{-0.8222} = \frac{-4.4228}{-0.8222} = 5.38$

$T = 5.38 \times 10^9$ years

0.2

0.4

(e) T is not $>> 4.50 \times 10^9$ years but is $> 0.71 \times 10^9$ years 0.1

We can insert the approximate value for T (call it $T^* = 5.38 \times 10^9$ years) in the $2^{T/4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{T/0.710}$ term). We now leave in the –1's, although the –1 on the right-hand side has little effect and may be omitted). 0.1

Either $\quad 0.0120((2^{T/0.710} - 1) = 2^{T^*/4.50} - 1$

$$2^{T/0.710} - 1 = \frac{2^{1.1956} - 1}{0.0120} = \frac{2.2904 - 1}{0.0120} = 107.5$$

$$T = 0.710 \frac{\log 108.5}{\log 2} = 4.80(0)$$

0.2

Put* T* = *4.80(0) × 10^9 years

$$2^{T/0.710} = \frac{2^{1.0668} - 1}{0.0120} = \frac{2.0948 - 1}{0.0120} = 91.2$$

$$T = 0.710 \frac{\log 91.2}{\log 2} = 4.62(3)$$

Further iteration gives 4.52

0.1

or

$0.0120(e^{0.9762T} - 1) = (e^{0.1540T^*} - 1)$ and similar

So more accurate answer for* T *to be in range 4.6 × 10^9 years to 4.5 × 10^9 years (either acceptable).

D Spherical charge

(a) Charge density = $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$ within sphere

0.3

$x \le R$ Field at distance x:

$$E = \frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x^2} = \frac{Qx}{4\pi\varepsilon_0 R^3}$$

0.3

$x > R$ Field at distance x from the centre: $E = \dfrac{Q}{4\pi\varepsilon_0 x^2}$

0.2

0.8

(b) Method 1

Energy density is $\frac{1}{2}\varepsilon_0 E^2$.

0.1

$x \le R$

Energy in a thin shell of thickness δx at radius x is given by

$$= \frac{1}{2}\varepsilon_0 E^2\ 4\pi x^2 \delta x = \frac{1}{2} 4\pi\varepsilon_0 \frac{Q^2 x^2}{(4\pi\varepsilon_0)^2 R^6} x^2\ \delta x$$ 0.1

Energy within the spherical volume = $\dfrac{1}{2}\dfrac{Q^2}{(4\pi\varepsilon_0)R^6}\displaystyle\int_{x=0}^{x=R} x^4 \mathrm{d}x = \dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

0.2

$x > R$

Energy within spherical shell = $\frac{1}{2}\varepsilon_0 E^2\ 4\pi x^2 \delta x = \frac{1}{2} 4\pi\varepsilon_0 \dfrac{Q^2}{(4\pi\varepsilon_0)^2 x^4} x^2\ \delta x$

0.1

Energy within the spherical volume for $x > R$

$$= \frac{1}{2}\frac{Q^2}{(4\pi\varepsilon_0)}\int_{x=R}^{x=\infty}\frac{1}{x^2}\mathrm{d}x = \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.2

Total energy associated with the charge distribution = $\dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$+ \dfrac{1}{8}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$$= \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.1

0.8

Method 2

A shell with charge $4\pi x^2 \delta x \rho$ moves from ∞ to the surface of a sphere radius x where the electric potential is 0.1

$$\frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x} = \frac{x^2 \rho}{3\varepsilon_0}$$

0.2

and will therefore gain electrical potential energy $\left(\frac{x^2\rho}{3\varepsilon_0}\right)\left(4\pi r^2 \rho\right)\delta x$

0.1

Total energy of complete sphere = $\int_{x=0}^{x=R} \frac{4\pi\rho^2 x^4}{3\varepsilon_0} dx = \frac{4}{15}\frac{\pi\rho^2 R^5}{\varepsilon_0}$

0.2

Putting Q = charge on sphere = $\frac{4}{3}\pi R^3 \rho$, $\rho = \frac{3Q}{4\pi R^3}$

So that total energy is = $\frac{4}{15}\pi\left(\frac{9Q^2}{16\pi^2 R^6}\right)\frac{R^5}{\varepsilon} = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}$

0.2

0.8

(c) Binding energy $E_{\text{binding}} = E_{\text{electric}} - E_{\text{nuclear}}$

0.1

Binding energy is a negative energy

Therefore -8.768 = E_{electric} - 10.980 MeV per nucleon

E_{electric} = 2.212 MeV per nucleon

0.1

Radius of cobalt nucleus is given by $R = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 E_{electric}^{total}}$

$$= \frac{3\times 27^2 \times (1.60\times 10^{-19})^2}{20\times\pi\times 8.85\times 10^{-12}\times 2.212\times 10^6 \times 57\times 1.60\times 10^{-19}} \text{ m}$$

= 5.0×10^{-15} m

0.2

0.4

E E.M. Induction

Method 1 Equating energy

Horizontal component of magnetic field B inducing emf in ring:

$B = 44.5 \times 10^{-6} \cos 64^{\circ}$ 0.2

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$

where a = radius of ring 0.1

Instantaneous emf = $\frac{d\phi}{dt} = B\pi a^2 \frac{d \sin \omega t}{dt}$ where ω = angular velocity

$= B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos\theta$ 0.1

R.m.s. emf over 1 revolution = $\frac{B\pi a^2\omega}{\sqrt{2}}$ ***0.2***

Average resistive heating of ring = $\frac{B^2\pi^2 a^4\omega^2}{2R}$ 0.1

Moment of inertia = $\frac{1}{2}ma^2$ 0.1

Rotational energy = $\frac{1}{4}ma^2\omega^2$ where m = mass of ring 0.1

Power producing change in ω = $\frac{d}{dt}\left\{\frac{1}{4}ma^2\omega^2\right\}$ =

$\frac{1}{4}ma^2 2\omega \frac{d\omega}{dt}$

0.1

Equating: $$\frac{1}{2}ma^2\omega \frac{d\omega}{dt} = -\frac{B^2\pi^2 a^4\omega^2}{2R}$$

0.1

$$\frac{d\omega}{\omega} = -\frac{B^2\pi^2 a^2}{mR}dt$$

0.1

If T is time for angular velocity to halve,

$$\int_{\omega}^{\omega/2}\frac{d\omega}{\omega} = -\int_0^T \frac{B^2\pi^2 a^2}{mR}dt$$

0.1

$$\ln 2 = \frac{B^2\pi^2 a^2}{mR}T$$ 0.2

But R = $\frac{2\pi a\rho}{A}$ where A is cross-sectional area of copper ring 0.1

$m = 2\pi ad A$ (d = density) 0.1

$$\ln 2 = \frac{B^2\pi^2 a^2 T}{\frac{2\pi a\rho}{A}2\pi adA} = \frac{B^2 T}{4\rho d}$$ 0.1

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^{3} \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{ s}$$

$= 1.10(2) \times 10^6$ s (=306 hr = 12 days 18 hr) 0.2

2.0

(Part E)

Method 2 Back Torque

Horizontal component of magnetic field = B = $44.5 \times 10^{-6} \cos 64^{o}$ 0.2

Cross-section of area of ring is A

Radius of ring = a

Density of ring = d

Resistivity = ρ

ω = angular velocity (ω positive when clockwise)

Resistance $R = \rho \dfrac{2\pi a}{A}$ 0.1

Mass of ring m = $2\pi a A d$ 0.1

Moment of inertia = $M = \dfrac{1}{2} m a^2$ 0.1

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$ 0.1

Instantaneous emf = $\dfrac{d\phi}{dt} = B\pi a^2 \dfrac{d \sin \omega t}{dt} = B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos\theta$ 0.1

Induced current = I = $B\pi a^2 \cos\theta / R$

Torque opposing motion = $(B\pi a^2 \cos\theta)\, I = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \cos^2\theta$ 0.1

Work done in small $\delta\theta = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \dfrac{1}{2}\left(\cos 2\theta + 1\right)\delta\theta$ 0.1

Average torque = (work done in 2π revolution)/2π

$$= \frac{1}{2\pi\, R}\left(B\pi\pi^{\,2}\right)^2 \omega \frac{1}{2} 2\pi = \frac{1}{2R}\left(B\pi\pi^{\,2}\right)^2 \omega$$ 0.1

This equals $M \dfrac{d\omega}{dt}$ so that $M \dfrac{d\omega}{dt} = -\dfrac{B(\pi a^2) B(\pi a^2)\frac{1}{2}}{(\rho / A)(2\pi a)} \omega$ 0.2

$$\frac{1}{2}(2\pi a A d) a^2 \frac{d\omega}{dt} = -\frac{B^2 (\pi a^2)^2 A}{4\rho\pi a}\omega$$

$$\frac{d\omega}{dt} = -\frac{B^2}{4\rho d}\omega$$ 0.2

$$\int_{\omega}^{\omega/2} \frac{d\omega}{\omega} = \int_0^T \frac{B^2}{4\rho d} dt$$ 0.2

$$\ln 2 = \frac{B^2 T}{4\rho d}$$ 0.2

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2}\,\text{s}$$

= $1.10(2) \times 10^6$ s =306 hr = 12 days 18 hr

0.2

2.0