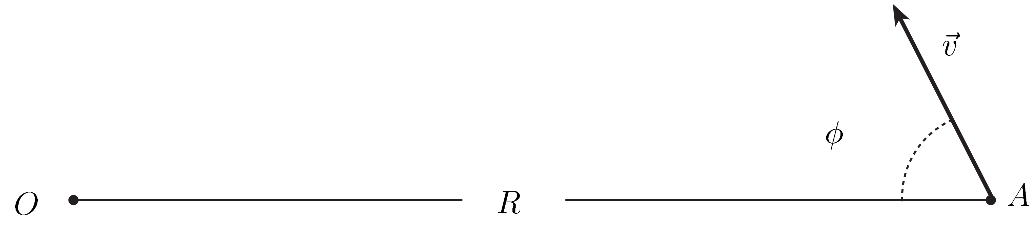


Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda = \lambda_0(1 - \beta\cos\phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\,\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\,. \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

time [days]	θ_1 [as]	θ_2 [as]
0	0.139	0.076
7	0.253	0.139
13	0.354	0.190
20	0.468	0.253
27	0.601	0.316
34	0.709	0.367

The uncertainty in the readings by the ruler is estimated to be ± 0.5 mm, resulting in the uncertainty of $\pm$ 0.013 as in the θ values. We plot the data in Figure 3.3.

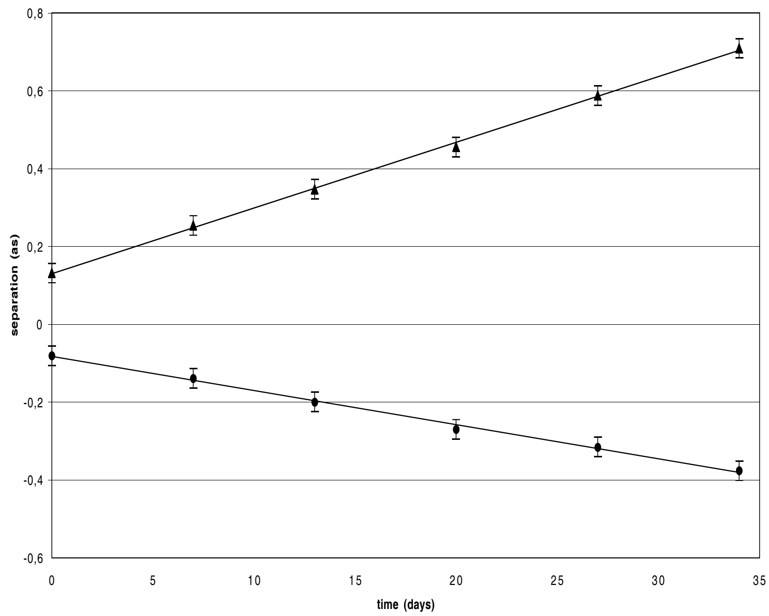


Figure 3.3: *The angular distances θ_1 and θ_2 (in as) as functions of the time in days.*

Fitting straight lines through the data results in:

$$\boldsymbol{\omega_1} = \mathrm{d}\theta_1/\mathrm{dt} = \mathbf{(17.0 \pm 1.0)\ mas/day} = 9.54 \cdot 10^{-13}\mathrm{rad/s} \tag{3.2}$$

$$\boldsymbol{\omega_2} = \mathrm{d}\theta_2/\mathrm{dt} = \mathbf{(8.7 \pm 1.0)\ mas/day} = 4.88 \cdot 10^{-13}\mathrm{rad/s} \tag{3.3}$$

$$\boldsymbol{v'_{1,\perp}} = \omega_1\, R = 9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19} \tag{3.4}$$

$$= \mathbf{3.68 \cdot 10^8 m/s \approx (1.23 \pm 0.07)}\ \boldsymbol{c} \tag{3.5}$$

$$\boldsymbol{v'_{2,\perp}} = \mathbf{1.89 \cdot 10^8 m/s \approx (0.63 \pm 0.07)}\ \boldsymbol{c} \tag{3.6}$$

b) We consider the motion of the source during the time interval Δt from the point A to the point A', see Figure 3.4.

We then have

$$\vec{r}_{AA'} = \vec{r}_{A'} - \vec{r}_A = \vec{v} \cdot \Delta t\ . \tag{3.7}$$

Now let $\Delta t'$ denote the difference in arrival times at O of the signals from A and A'. Due to the different distances to A and A' and the finite speed of light, c, we have

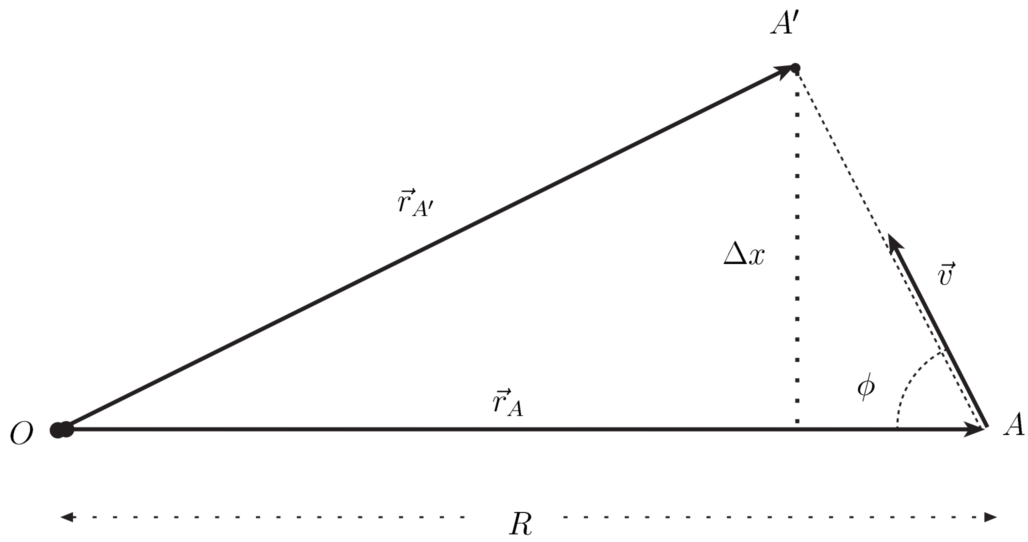


Figure 3.4: *The observer is at O and the original position of the source is at A. The velocity vector is $\vec{v}$.*

$$\Delta t' = \Delta t + (r_{A'} - r_A)/c \,. \tag{3.8}$$

For small Δt, such that $v\ \Delta t << r_A = R$, we have

$$r_{A'} - r_A \approx -v\ \Delta t \cos\phi \tag{3.9}$$

and hence

$$\Delta t' \approx \Delta t\ (1 - \beta\cos\phi)\ ;\ \beta = v/c\,. \tag{3.10}$$

This implies that an observer at O will find the apparent transverse speed of the source to be

$$\boldsymbol{v'_\perp} = \frac{\Delta x}{\Delta t'} = \frac{\Delta x}{\Delta t\ (1-\beta\cos\phi)} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{1-\beta\cos\phi}} \tag{3.11}$$

where we have used that the real transverse speed in the reference frame of the observer is $v_\perp = \Delta x/\Delta t = c\beta\sin\phi$.

The angular speed observed at O is

$$\boldsymbol{\omega} = \frac{v'_\perp}{R} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{R\ (1-\beta\cos\phi)}} \tag{3.12}$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi = \phi_1$ we have $\sin\phi_2 = \sin\phi$ and $\cos\phi_2 = -\cos\phi$. Equation (3.12) then gives:

$$\omega_1 = \frac{\beta\ c\ \sin\phi}{R\ (1-\beta\cos\phi)} \tag{3.13}$$

$$\omega_2 = \frac{\beta\ c\ \sin\phi}{R\ (1+\beta\cos\phi)}\,. \tag{3.14}$$

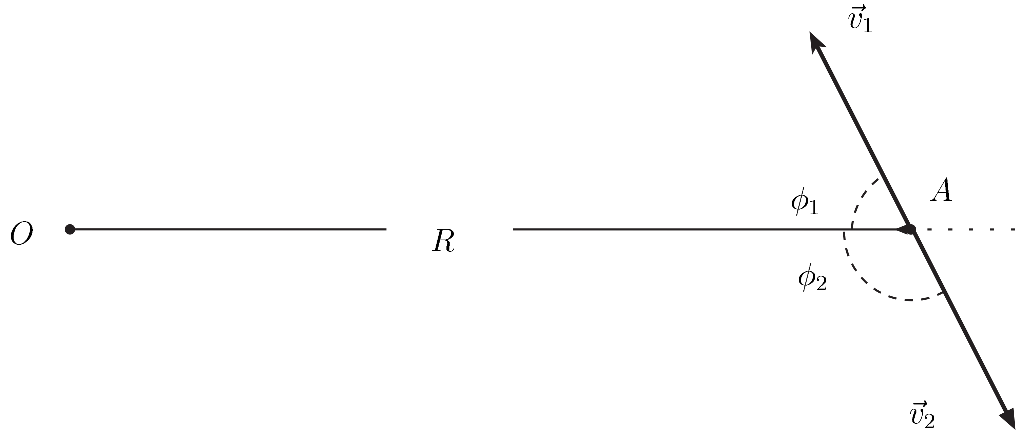


Figure 3.5: *If the two objects have equal speeds but opposite velocities we have* $v_1 = v_2 = v$, $\beta_1 = \beta_2 = \beta$ *and* $\phi_2 = \pi - \phi_1$.

The quantities ω_1, ω_2 and R are given, but β and ϕ are to be determined as stated in the problem text. Simple algebra gives:

$$(1 - \beta\ \cos\phi)\ \omega_1\ \omega_2 \quad = \quad \beta\ c\ \sin\phi\ \omega_2/R \tag{3.15}$$
$$(1 + \beta\ \cos\phi)\ \omega_2\ \omega_1 \quad = \quad \beta\ c\ \sin\phi\ \omega_1/R\ . \tag{3.16}$$

Subtracting (3.15) from (3.16) gives:

$$2\ \beta\ \cos\phi\ \omega_2\ \omega_1 = \beta\ c\ \sin\phi\ (\omega_1 - \omega_2)/R \tag{3.17}$$

$$\tan\phi = \frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)} \tag{3.18}$$

$$\boldsymbol{\phi = \arctan\left(\frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)}\right)}\ . \tag{3.19}$$

Dividing (3.15) by (3.16) gives β in terms of $\cos\phi$ and the known quantities ω_1 and ω_2:

$$\omega_1\ (1 - \beta\ \cos\phi) = \omega_2\ (1 + \beta\ \cos\phi) \tag{3.20}$$

$$\boldsymbol{\beta = \frac{\omega_1 - \omega_2}{\cos\phi\ (\omega_1 + \omega_2)}}\ . \tag{3.21}$$

Inserting the values of ω_1 and ω_2 from part (a) and the given values of R and c we get:

$$\boldsymbol{\phi} \quad = \quad \arctan(2.57) = \mathbf{1.20\ rad = 68.8^\circ \pm 2^\circ} \tag{3.22}$$
$$\boldsymbol{\beta} \quad = \quad \mathbf{0.892 \pm 0.08} \tag{3.23}$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$\frac{\beta \sin\phi}{1-\beta\cos\phi} \geq 1. \tag{3.24}$$

If $\beta < 1$ condition (3.24) is equivalent to:

$$\begin{aligned} \beta \sin\phi &\geq 1-\beta\cos\phi && (3.25)\\ \beta\,(\sin\phi+\cos\phi) &\geq 1 && (3.26)\\ \beta\,\sqrt{2}\left(\sin\phi\cos\frac{\pi}{4}+\cos\phi\sin\frac{\pi}{4}\right) &\geq 1 && (3.27)\\ \sin\left(\phi+\frac{\pi}{4}\right) &\geq \frac{1}{\beta\sqrt{2}} && (3.28) \end{aligned}$$

and hence (3.24) is satisfied if:

$$\beta > \boldsymbol{f(\phi)} = \left(\boldsymbol{\sqrt{2}\sin(\phi+\pi/4)}\right)^{\boldsymbol{-1}}. \tag{3.29}$$

The physically relevant region in the (β, ϕ)-plane is:

$$\boldsymbol{(\beta,\phi) \in [0,1[\times[0,\pi]}\ . \tag{3.30}$$

It is obvious that (3.24) can only be satisfied for $\phi \in [0, \pi/2]$ and (3.28) can only have a solution for ϕ if $\beta \geq 1/\sqrt{2}$.

We therefore take a closer look at the region

$$(\beta,\phi)\ \in\ [2^{-1/2}, 1[\ \times\ [0, \pi/2] \tag{3.31}$$

The mapping

$$(\beta,\phi) \mapsto \beta\sin\left(\phi+\frac{\pi}{4}\right) \tag{3.32}$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

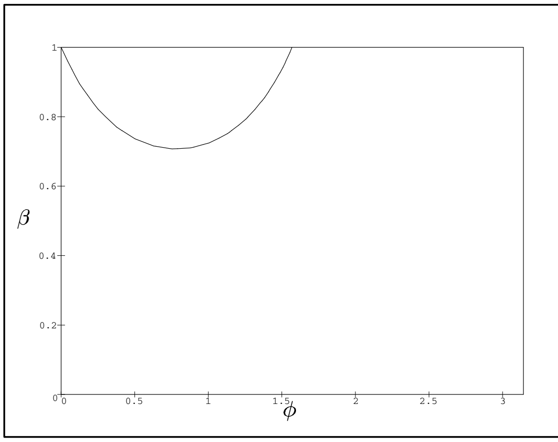
$$\beta\sin\left(\phi+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{3.33}$$

This defines β as a function of ϕ which is shown in Figure 3.6 as the curve bounding the shaded area where $v'_\perp > c$.

e) To find the extrema of $v'_\perp$ as a function of ϕ we differentiate (3.11) and get

$$\frac{d}{d\phi}\left(\frac{v'_\perp}{c}\right) = \frac{\beta(\cos\phi-\beta)}{(1-\beta\cos\phi)^2}\ . \tag{3.34}$$

This is zero for $\phi = \phi_m$ where:



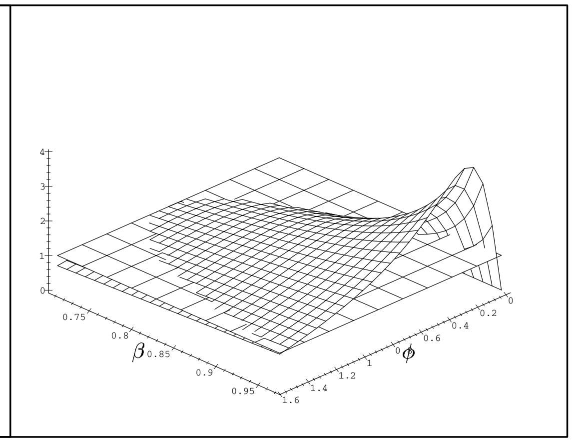


Figure 3.6: *The region between the horizontal line and the curve in the upper left hand corner shows where* $v'_\perp/c > 1$.

Figure 3.7: *The curved surface is* $v'_\perp/c$ *as a function of* β *and* ϕ. *The plane represents the constant function* $\beta = 1$.

$$\cos\phi_m = \beta\ ;\ \phi_m = \arccos\beta \in]0, \pi/2] \tag{3.35}$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right) = -\beta\left(\frac{\sin\phi}{(1-\beta\cos\phi)^2} + 2\frac{\beta\sin\phi(\cos\phi-\beta)}{(1-\beta\cos\phi)^3}\right) \tag{3.36}$$

At the extremum

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right)\Bigg|_{\phi_m} = -\frac{\beta\sin\phi_m}{(1-\beta^2)^2} < 0 \tag{3.37}$$

showing that ϕ_m corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$\boldsymbol{(v'_\perp)_{max} = \frac{\beta c}{\sqrt{1-\beta^2}}} \tag{3.38}$$

From this and (3.35) we see that

$$(v'_\perp)_{max} \underset{\beta\to 1}{\longrightarrow} \infty\ ;\ \phi_m \underset{\beta\to 1}{\longrightarrow} 0\ . \tag{3.39}$$

Figure 3.7 shows $v'_\perp/c$ as a function of β and ϕ in the region $(\beta, \phi) \in [2^{-1/2}, 1[\times [0, \pi/2]$.

f) We have the equations for relativistic Doppler-shift:

$$\frac{\lambda_{1,2}}{\lambda_0} = \frac{1 \mp \beta \cos\phi}{\sqrt{1-\beta^2}} \tag{3.40}$$

We add them, define an auxiliary ratio ρ and solve for β.

$$\rho := \frac{\lambda_1 + \lambda_2}{2\ \lambda_0} = \frac{1}{\sqrt{1-\beta^2}} \tag{3.41}$$

$$\rho^2\ (1-\beta^2) = 1 \tag{3.42}$$

$$\beta = \sqrt{1 - 1/\rho^2} = \sqrt{1 - \frac{4\ \lambda_0^2}{(\lambda_1+\lambda_2)^2}} \tag{3.43}$$

giving

$$\boldsymbol{\alpha = 4} \tag{3.44}$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns β, ϕ and R. For instance, we may calculate β from (3.43), insert that into (3.21), and solve for ϕ. The distance R can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that ω_1 and ω_2 are known.

3.3 Grading scheme

Part 1(a)	
Answer i): equation (3.2), ω_1 in the range (16.5-17.5) mas/day	**0.8**
Answer ii): equation (3.3), ω_2 in the range (8.2-9.2) mas/day	**0.8**
Answer iii): equation (3.4), for $v'_{1,\perp}$ in the range (1.13-1.30)c	**0.2**
Answer iv): equation (3.6), for $v'_{2,\perp}$ in the range (0.56-0.70)c	**0.2**

Part 1(b)	
Answer i): $v'_\perp(\beta,\phi)$, equation (3.11)	**2.5**
Answer ii): $\omega(\beta,\phi)$, equation (3.12)	**0.5**

Part 1(c)	
Answer i): $\phi(\omega_1,\omega_2)$, equation (3.19)	**0.3**
Answer ii): $\beta(\omega_1,\omega_2)$, equation (3.21)	**0.3**
Answer iii): ϕ numerical in the range 67° - 71°	**0.2**
Answer iv): β numerical in the range 0.81-0.97	**0.2**

Part 1(d)	
Answer i): Condition $\beta > f(\phi)$, equation (3.29)	**1.0**
Answer ii): Condition on (β,ϕ), graph	**1.0**

Part 1(e)	
Answer: $(v'_\perp)_{max}$, equation (3.38)	**1.0**

Part 1(f)	
Answer: β in terms of λ-s, by α, equation (3.44)	**1.0**