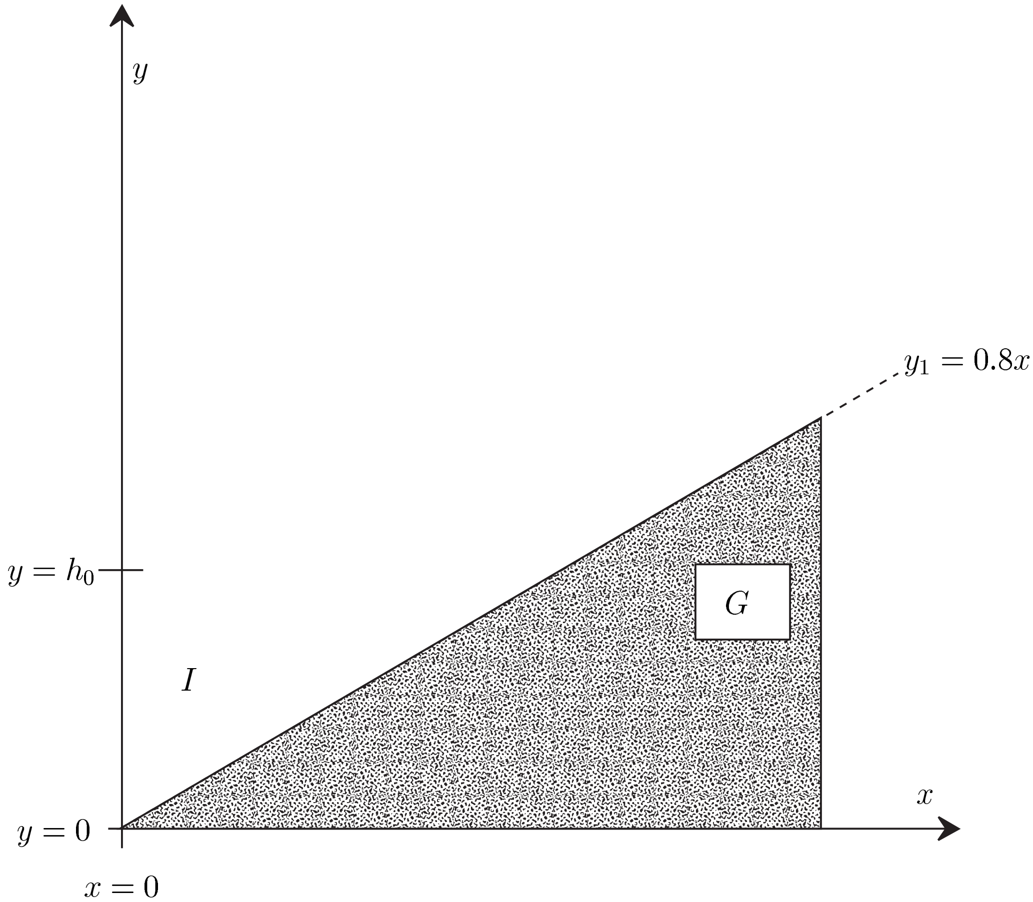

Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

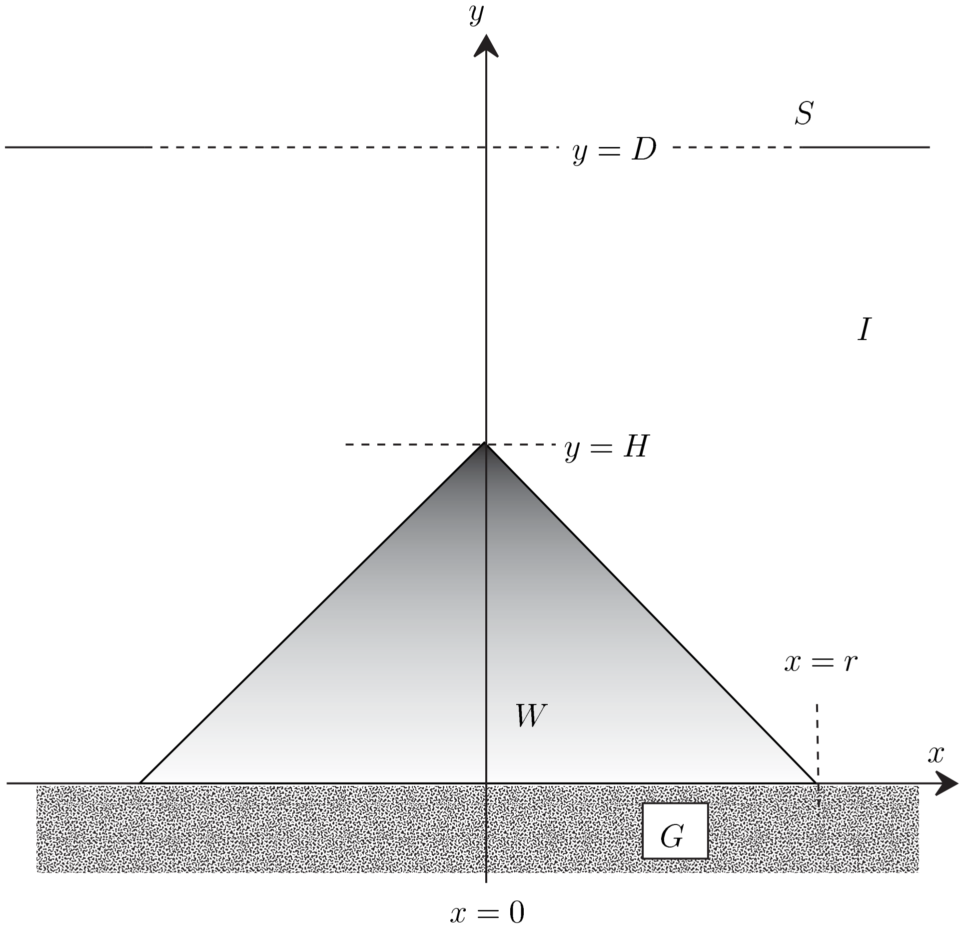

Figure 2.3: *A vertical section through the mid-plane of a water cone inside an ice cap. S: surface, W: water, G: ground, I: ice cap.*

$$\boldsymbol{d} = \frac{J_Q \cdot 1 \text{ year}}{L_i\ \rho_i} = \frac{0.06 \text{ J s}^{-1} \text{ m}^{-2}\ 365.25\ \cdot 24\ \cdot 60\ \cdot 60 \text{ s}}{3.4 \cdot 10^5 \text{ J/kg } 917 \text{ kg/m}^3} = \mathbf{6.1 \cdot 10^{-3} m} \tag{2.3}$$

b)

Let p_a be the atmospheric pressure, taken to be constant. At a depth z inside the ice cap the pressure is given by:

$$p = \rho_i g z + p_a \tag{2.4}$$

Therefore, at the bottom of the ice cap, where $z = y_2 - y_1$:

$$\boldsymbol{p} = \rho_i g(y_2 - y_1) + p_a \tag{2.5}$$
$$= \boldsymbol{\rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a} \tag{2.6}$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.

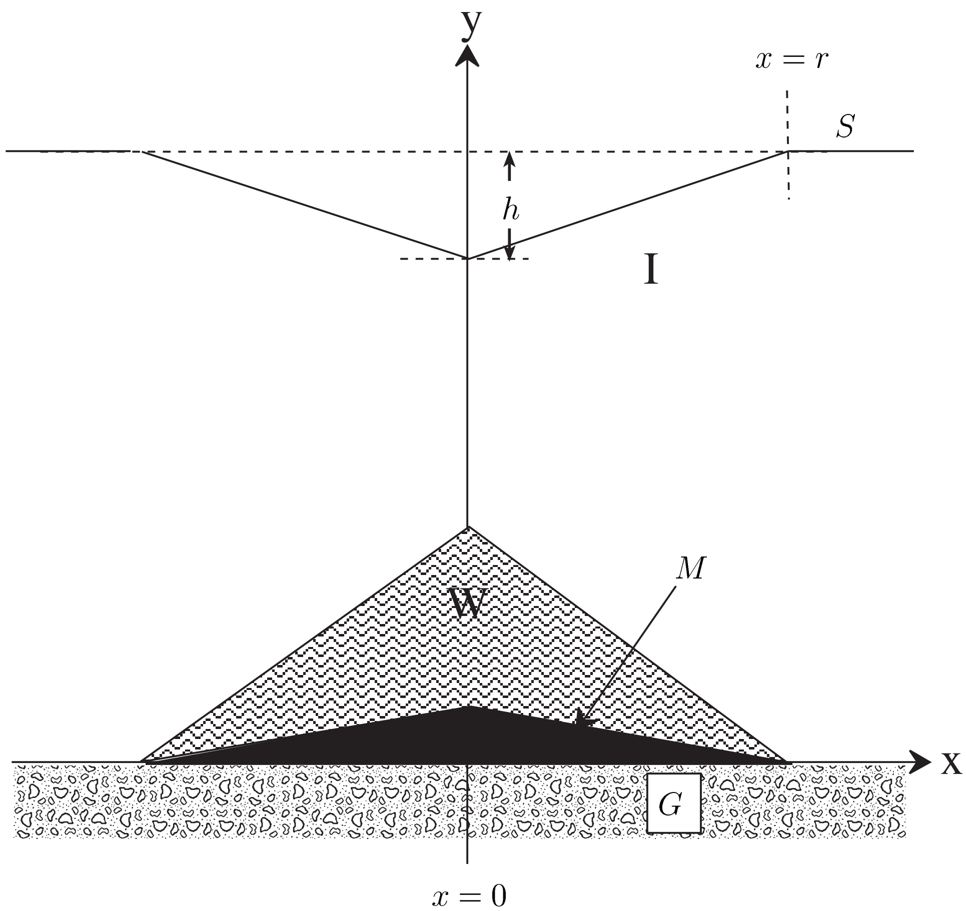

Figure 2.4: *A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, G: ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.*

$$p = constant - \rho_w g y_1 \tag{2.7}$$
$$= constant - \rho_w g x \tan\alpha \tag{2.8}$$

Therefore

$$\rho_i g x(\tan\beta - \tan\alpha) = -\rho_w g x \tan\alpha \tag{2.9}$$

leading to

$$\mathbf{tan}\,\boldsymbol{\beta} = -\frac{\rho_w - \rho_i}{\rho_i}\ \tan\alpha = \boldsymbol{-}\frac{\boldsymbol{\Delta\rho}}{\boldsymbol{\rho_i}}\ \mathbf{tan}\,\boldsymbol{\alpha} \approx -0.091\ \tan\alpha \tag{2.10}$$
$$\boldsymbol{s = -\Delta\rho/\rho_i = -0.091} \tag{2.11}$$
$$\tag{2.12}$$

where the minus-sign is significant.

This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. – We now proceed with the solution.

With $\tan\alpha = 0.8$, we get $\tan\beta = -0.073$ and

$$\boldsymbol{y_2 = 2\ \textbf{km} - 0.073\ x} \tag{2.13}$$

The students are supposed to draw this line on a graph.

c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$\begin{aligned} h &= |r \tan\beta| = \frac{\Delta\rho}{\rho_i} r \tan\alpha && (2.14) \\ &= \frac{\Delta\rho}{\rho_i} H && (2.15) \\ &= 0.091 \ \cdot 1 \text{ km} = 91 \text{ m}. && (2.16) \end{aligned}$$

The students are supposed to show this result as a graph.

d)

The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$. We assume that the height of the intrusion is h_1. We may say that it firstly melts an ice cone of its own volume $V_1 = \frac{1}{3}\pi r^2 h_1$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to 0°C the intrusion will further melt a volume equivalent to a cone of height h_3, its water accumulating in place, forming a cone of height $h_3' = \frac{\rho_i}{\rho_w} h_3$ relative to the top of the intrusion. The total height of the ice cone melted is

$$h_{tot} = h_1 + h_2 + h_3 \tag{2.17}$$

The depth of the depression at the surface will be given by

$$h = \frac{\Delta\rho}{\rho_i} (h_1 + h_3') \tag{2.18}$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$\boldsymbol{H} = h_1 + h_3' = \frac{\rho_i}{\Delta\rho} h = \mathbf{1.1 \times 10^3 m} \tag{2.19}$$

The heat balance gives

$$\frac{1}{3} \pi r^2 \left\{ \rho_r h_1 (L_r + c_r \Delta T) - \rho_i L_i h_{tot} \right\} = 0 \tag{2.20}$$

where $\Delta T = 1200°C$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ and $h_3 = \frac{\rho_w}{\rho_i} h'_3$ we obtain

$$h_{tot} = h_1 + \frac{\Delta\rho}{\rho_i} h_1 + \frac{\rho_w}{\rho_i} h'_3 = \frac{\rho_w}{\rho_i} (h_1 + h'_3) \tag{2.21}$$

Therefore (using equation (2.19))

$$h_{tot} = \frac{\rho_w}{\rho_i} (h_1 + h'_3) = \frac{\rho_w}{\rho_i} H = \frac{\rho_w}{\Delta\rho} h = 1.20 \cdot 10^3 \text{m} \tag{2.22}$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for h_1:

$$\rho_r \, h_1 \, (L_r + \, c_r \, \Delta T) = \frac{\rho_i \, \rho_w \, L_i \, h}{\Delta\rho} \tag{2.23}$$

$$\boldsymbol{h_1} = \frac{\rho_i \, \rho_w \, L_i \, h}{\Delta\rho \, \rho_r \, (L_r + \, c_r \, \Delta T)} \tag{2.24}$$

$$= \mathbf{103 \text{ m}} \tag{2.25}$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$\boldsymbol{m_{tot}} = \rho_i \, (1/3) \, \pi \, r^2 \, h_{tot} = \mathbf{2.9 \cdot 10^{11} \text{ kg}} \tag{2.26}$$

The mass of the water which flows away is

$$\boldsymbol{m'} = \frac{h_1 + h_2}{h_{tot}} m_{tot} = \frac{\rho_w \, h_1}{\rho_i \, h_{tot}} m_{tot} = \mathbf{2.7 \cdot 10^{10} \text{ kg}} \tag{2.27}$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

2.3 Grading scheme

2(a)	
Answer: equation (2.3), $d = 6.1 \cdot 10^{-3}$ m	**0.5**
2(b)	
Answer i): equation (2.6): $p = \rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a$	**1.0**
Answer ii): equation (2.10): $s = -\frac{\rho_w - \rho_i}{\rho_i} = -\frac{\Delta\rho}{\rho_i}$	**2.0**
Answer iii): Graph based on equation (2.13)	**0.5**
2(c)	
Answer: Depth, radius and graph, $r = 1000$ m, $h = 91$ m	**1.0**
2(d)	
Answer i): Height of water cone as in (2.19): $H = 1.1 \cdot 10^3$ m	**2.0**
Answer ii): Height of intrusion as in (2.25): $h_1 = 103$ m	**1.0**
Answer iii): Total mass of melt water as in (2.26): $m_{tot} = 2.9 \cdot 10^{11}$ kg	**0.5**
Answer iv): Mass of water flowing away as in (2.27): $m' = 2.7 \cdot 10^{10}$ kg	**1.0**
Answer v): Graph	**0.5**