# 2 Water under an ice cap[6]

## 2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

| | |
|---|---|
| Density of water: | $\rho_w = 1.000 \cdot 10^3$ kg/m$^3$ |
| Density of ice: | $\rho_i = 0.917 \cdot 10^3$ kg/m$^3$ |
| Specific heat of ice: | $c_i = 2.1 \cdot 10^3$ J/(kg °C) |
| Specific latent heat of ice: | $L_i = 3.4 \cdot 10^5$ J/kg |
| Density of rock and magma: | $\rho_r = 2.9 \cdot 10^3$ kg/m$^3$ |
| Specific heat of rock and magma: | $c_r = 700$ J/(kg °C) |
| Specific latent heat of rock and magma: | $L_r = 4.2 \cdot 10^5$ J/kg |
| Average outward heat flow through the surface of the earth: | $J_Q = 0.06$ W/m$^2$ |
| Melting point of ice: | $T_0 = 0$°C, constant |

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness $d$ of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle $\alpha$. The upper surface of the cap slopes by an angle $\beta$ as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is $h_0$. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan\alpha, \quad y_2 = h_0 + x \tan\beta \tag{2.1}$$

Derive an expression for the pressure $p$ at the bottom of the ice cap as a function of the horizontal coordinate $x$ and write it on the answer sheet.

Formulate mathematically a condition between $\beta$ and $\alpha$, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan\beta = s\tan\alpha$. Find the coefficient $s$ and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8\,x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness $h_0$ at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line $y_1$ and add a line $y_2$ showing the upper surface of the ice. Indicate on the figure which line is which.

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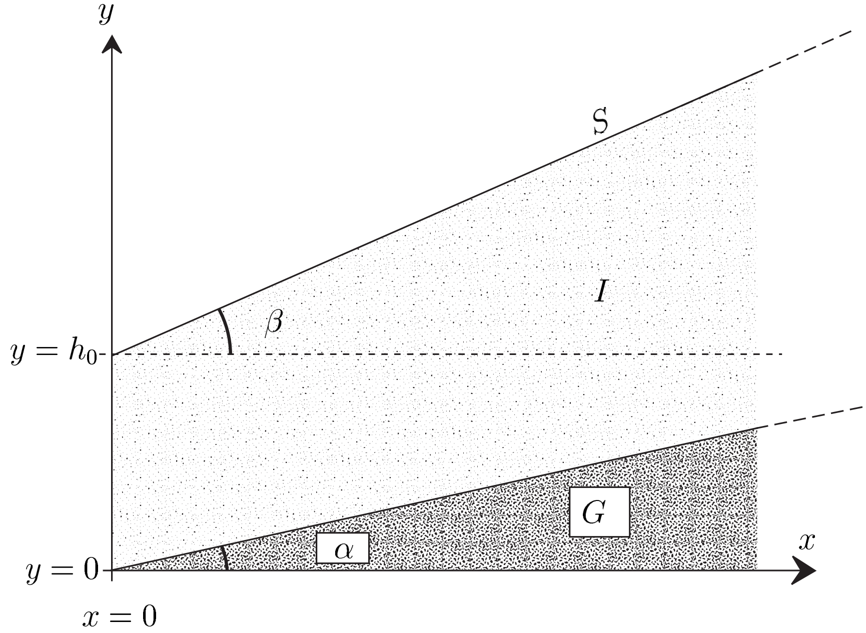


Figure 2.1: *Cross section of an ice cap with a plane surface resting on an inclined plane ground. S: surface, G: ground, I: ice cap.*

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth $h$ of 100 m and a radius $r$ of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C. Subsequently,

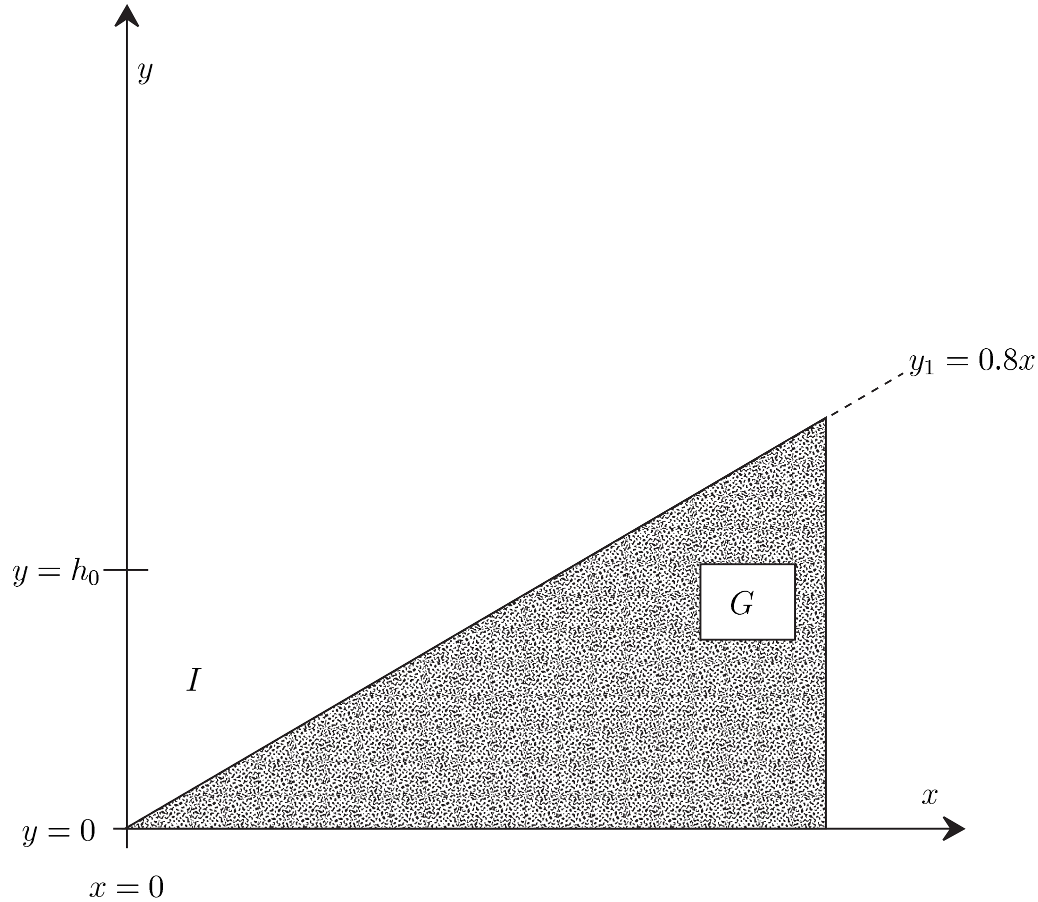


Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height $H$ of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height $h_1$ of the intrusion.
3. The total mass $m_{tot}$ of the water produced and the mass $m'$ of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

## 2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$