Theory Question 1: Solutions

Scaling

(a) Let the original spring have length l and spring constant k. The frequency f of a mass m oscillating on the end of this spring is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The spring constant k means that a force F is required to produce an extension Δx:

$$k = \frac{F}{\Delta x}$$

Consider the mid-point of the spring during such an extension; it has only moved a distance $\Delta x/2$, while experiencing the same force F. Therefore the spring constant of one half of the spring is given by:

$$k' = \frac{F}{\Delta x / 2} = 2k$$

The frequency of the mass on the half-spring is:

$$f' = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \underline{\sqrt{2} f}$$

Method 1 using angular momentum quantization:

The de Broglie wavelength of the particle is:

$$\lambda = \frac{h}{p}.$$

By de Broglie principle (ground state):

$$2\pi\ r = \lambda$$

Thus giving the result

$$pr = mvr = \hbar$$

(This can be stated directly as quantization of angular momentum.)

Using the Bohr model, we can consider a centripetal force due to electrostatic attraction:

$$\frac{mv^2}{r} = \frac{ke^2}{r^2}, \quad \therefore\ v = \sqrt{\frac{ke^2}{mr}}$$

$$\sqrt{k\ e^2 mr} = \hbar; \quad i.e.\ r \propto \frac{1}{m}$$

The radius of the muonic hydrogen atom is given by:

$$a_\mu = \frac{a_0}{207} = \underline{0.256\ \text{pm}}$$

Method 2 using Dimensional Analysis:

The radius r of the hydrogen atom in ground state depends on the following quantities:

- Mass m of the orbiting particle. (Since the mass of the nucleus is assumed to be much larger than the orbiting particle's mass, the nucleus can be regarded as stationary, and thus the atomic radius does not depend on the nuclear mass.)
- Electrical force between the orbiting particle and the nucleus. This depends on the charge of the nucleus q_n, the charge of the orbiting particle q and the constant ε_0.
- $\hbar$. This is because the angular momentum is quantized as demonstrated above.

Thus:

$$r = A\hbar^\alpha m^\beta q_n^{\gamma_1} q^{\gamma_2} \varepsilon_0{}^\delta$$

where A, α, β, γ_1, γ_2 and δ are dimensionless constants. The dimensional equation is:

$$[D]=[M]^{\alpha+\beta-\delta}[D]^{2\alpha-3\delta}[Q]^{\gamma_1+\gamma_2+2\delta}[T]^{2\delta-\alpha}$$

where [D] are distance dimensions, [M] are mass dimensions, [Q] are charge dimensions and [T] are time dimensions.

Thus, letting $\gamma=\gamma_1+\gamma_2$:

$$\begin{aligned}\alpha+\beta-\delta&=0\\ 2\alpha-3\delta&=1\\ \gamma+2\delta&=0\\ 2\delta-\alpha&=0\end{aligned}$$

Solving:

$$\begin{aligned}\alpha&=2\\ \beta&=-1\\ \gamma&=-2\\ \delta&=1\end{aligned}$$

This indicates that the radius is inversely proportional to the mass of the orbiting particle:

$$r\propto\frac{1}{m}$$

The radius of the muonic hydrogen atom is given by:

$$a_\mu=\frac{a_0}{207}=\underline{0.256\text{ pm}}$$

(c) If the solar power output is P and the radius of the earth's orbit is R, then T is given by equating incoming and outgoing radiation:

$$(1-r)\frac{P}{4\pi R^2}\cdot \pi R_E^2 = 4\pi R_E^2\, \varepsilon \sigma T^4$$

Where r is the reflectance of the earth with respect to solar radiation (albedo), R_E is the earth's radius, ε its emissivity and σ Stefan's constant. The solar power output is P and the mean orbital radius of the earth R. The emissivity is a function of temperature (not known *a priori*) but the change in temperature is expected to be small.

Plainly, $T \propto \sqrt{\frac{1}{R}}$, therefore a 1% reduction in R gives a 0.5% rise in T, i.e. 1.4 K

$$\underline{T' = 288.4\ \text{K}}$$

(d) Ideal gas equation for N molecules: $pV = NkT$. Two identical volumes of gas at the same pressure and temperature contain the same number of molecules; therefore the density of each is proportional to the mean molecular mass of the gas therein.

Here we use subscripts *d, m* and *w* to denote “dry”, “moist” and “water”.

For dry air, with mean molecular mass m_d:

$$\rho = \rho_d = m_d \frac{N_d}{V} = \frac{m_d p}{kT}$$

For moist air, with mean molecular mass m_m:

$$\rho_m = m_m \frac{N_m}{V} = \frac{m_m p}{kT}$$

For a mass M of dry air:

$$N_d \propto \frac{M}{28.8}$$

For a mass M' of moist air:

$$N_m \propto 0.02\frac{M'}{18} + 0.98\frac{M'}{28.8}$$

$$N_m = N_d$$

$$\frac{\rho_m}{\rho_d} = \frac{M'}{M} = \frac{1}{28.8\left(\frac{0.02}{18} + \frac{0.98}{28.8}\right)} = 0.9881$$

$$\rho' = \rho_m = 0.9925\rho_d = \underline{1.2352 \text{ kg/m}^3}$$

(e) The mechanical power P required for a helicopter to hover equals the downward thrust T of the rotor blades (equal to its weight W) times the mean velocity v of the downward moving column of air beneath its rotor blades:

$$P = Tv$$

The blades impart a velocity v to the air flowing past at a rate of dm/dt, and the swept area of the blades is A:

$$T = v\frac{dm}{dt}\,; \quad \frac{dm}{dt} = \rho A v\,; \quad \therefore\ W = T = \rho A v^2$$

If the size of the helicopter is characterized by a linear dimension L :

$$W \propto L^3\,;\ A \propto L^2$$

$$v \propto \sqrt{\frac{W}{A}} \propto \sqrt{L}\ , \quad \therefore\ P = Wv \propto L^{3.5}$$

Hence the power required for a half-scale helicopter is $\underline{P' = 0.0884P}$.
