Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{4.63 \cdot 10^6\,\mathrm{m},} \tag{1}$$

less than R, and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67 \cdot 10^{-6}\,\mathrm{s}^{-1}}}. \tag{2}$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2} m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

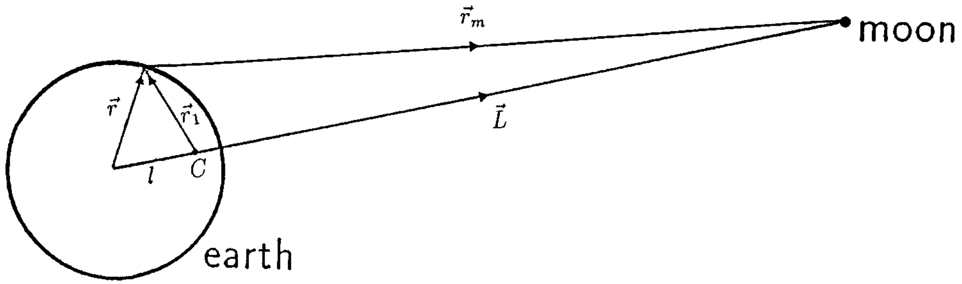
(3) Gravitational attraction to the moon,

$$-G\frac{mM_m}{|\vec{r}_m|},$$

where $\vec{r}_m$ is the distance from the moon.

Describing the position of m by polar coordinates r; ϕ in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^{\,2} = (\vec{r}-\vec{l}\,)^2 = r^2 - 2rl\cos\phi + l^2.$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\phi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{|\vec{r}_m|}. \tag{3}$$

Here l is given by (1) and

$$|\vec{r}_m| = \sqrt{(\vec{L}-\vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1+(r/L)^2 - 2(r/L)\cos\phi}.$$

c) Since the ratio $r/L = a$ is very small, we may use the expansion

$$\frac{1}{\sqrt{1+a^2-2a\cos\phi}} = 1 + a\cos\phi + a^2\,\tfrac{1}{2}(3\cos^2\phi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r,\phi)/m = -\tfrac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \tag{4}$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\phi - GmM_m\frac{r}{L^2}\cos\phi = 0,$$

when the value of ω_2, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy *V everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide h is much smaller than R, we have approximately

$$\frac{1}{r} = \frac{1}{R+h} = \frac{1}{R} \cdot \frac{1}{1+(h/R)} \cong \frac{1}{R}(1 - \frac{h}{R}) = \frac{1}{R} - \frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of ω into (4), we have

$$V(r,\phi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \quad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M}\left(\frac{R}{L}\right)^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, i.e.,

$$h = \frac{M_m r^2 R^2}{2ML^3}(3\cos^2\phi - 1),$$

then the mass point m has the same energy everywhere on the surface. Here r^2 can safely be approximated by R^2, giving the tidal bulge

$$\underline{\underline{h = \frac{M_m R^4}{2ML^3}(3\cos^2\phi - 1).}}$$

The largest value $h_{\max} = M_m R^4 / ML^3$ occurs for $\phi = 0$ or π, in the direction of the moon or in the opposite direction, while the smallest value

$$h_{\min} = -M_m R^4 / 2ML^3$$

corresponds to $\phi = \pi/2$ or $3\pi/2$.

The difference between high tide and low tide is therefore

$$h_{\max} - h_{\min} = \underline{\frac{3M_m R^4}{2ML^3}} = \underline{\underline{0.54\text{m.}}}$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)

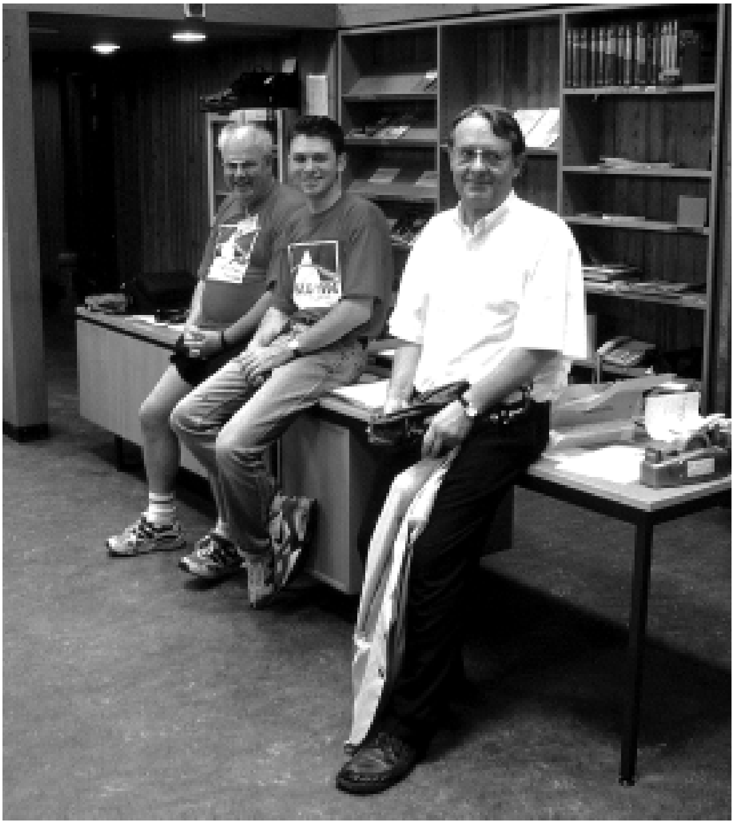


Photo: Arnt Inge Vistnes

Here we see the Exam Officer, Michael Peachey (in the middle), with his helper Rod Jory (at the left), both from Australia, as well as the Chief examiner, Per Chr. Hemmer. The picture was taken in a silent moment during the theory examination. Michael and Rod had a lot *of experience from the 1995 IPhO in Canberra, so their help was very effective and highly appreciated!*