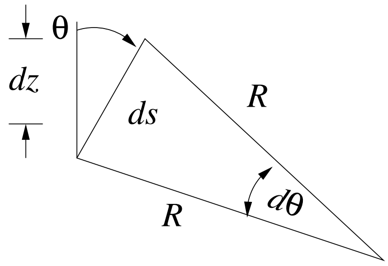
Solutions to Theoretical Question 2

(a)

Snell's Law may be expressed as

$$\frac{\sin\theta}{\sin\theta_0} = \frac{c}{c_0} \quad , \tag{1}$$

where c is the speed of sound.
Consider some element of ray path ds and treat this as, locally, an arc of a circle of radius R. Note that R may take up any value between 0 and ∞. Consider a ray component which is initially directed upward from S.



In the diagram, $ds = Rd\theta$, or $\dfrac{ds}{d\theta} = R$.

From equation (1), for a small change in speed dc,

$$\cos\theta d\theta = \frac{\sin\theta_0}{c_0} dc$$

For the upwardly directed ray $c = c_0 + bz$ so $dc = bdz$ and

$$\frac{\sin\theta_0}{c_0} b\, dz = \cos\theta d\theta \ , \quad \text{hence} \quad dz = \frac{c_0}{\sin\theta_0}\frac{1}{b}\cos\theta\, d\theta \quad .$$

We may also write (here treating ds as straight) $dz = ds\cos\theta$. So

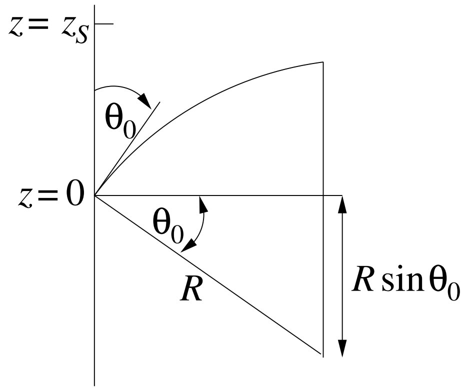
$$ds = \frac{c_0}{\sin\theta_0}\frac{1}{b}\, d\theta$$

Hence

$$\frac{ds}{d\theta} = R = \frac{c_0}{\sin\theta_0}\frac{1}{b} \quad .$$

This result strictly applies to the small arc segments ds. Note that from equation (1), however, it also applies for all θ, i.e. for all points along the trajectory, which therefore forms an arc of a circle with radius R until the ray enters the region $z < 0$.

(b)



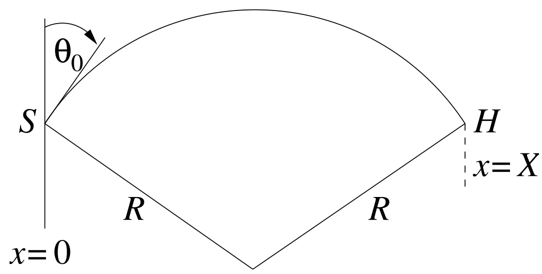
Here

$$\begin{aligned} z_s &= R - R\sin\theta_0 \\ &= R(1-\sin\theta_0) \\ &= \frac{c_0}{b\sin\theta_0}(1-\sin\theta_0) \quad , \end{aligned}$$

from which

$$\theta_0 = \sin^{-1}\left[\frac{c_0}{bz_s + c_0}\right] \quad .$$

(c)



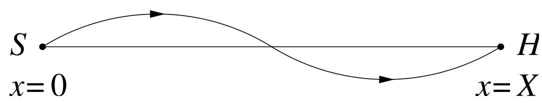
The simplest pathway between S and H is a single arc of a circle passing through S and H. For this pathway:

$$X = 2R\cos\theta_0 = \frac{2c_0\cos\theta_0}{b\sin\theta_0} = \frac{2c_0}{b}\cot\theta_0 \ .$$

Hence

$$\cot\theta_0 = \frac{bX}{2c_0} \quad .$$

The next possibility consists of two circular arcs linked as shown.



For this pathway:

$$\frac{X}{2} = 2R\cos\theta_0 = \frac{2c_0}{b}\cot\theta_0 \ .$$

i.e.

$$\cot\theta_0 = \frac{bX}{4c_0} \quad .$$

In general, for values of $\theta_0 < \frac{\pi}{2}$, rays emerging from S will reach H in n arcs for launch angles given by

$$\theta_0 = \cot^{-1}\left[\frac{bX}{2nc_0}\right] = \tan^{-1}\left[\frac{2nc_0}{bX}\right]$$

where $n = 1, 2, 3, 4, \ldots$
Note that when $n = \infty$, $\theta_0 = \frac{\pi}{2}$ as expected for the axial ray.

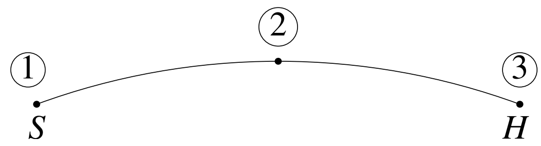
(d)

With the values cited, the four smallest values of launch angle are

n	θ_0 (degrees)
1	86.19
2	88.09
3	88.73
4	89.04

(e)

The ray path associated with the smallest launch angle consists of a single arc as shown:



We seek

$$\int_1^3 dt = \int_1^3 \frac{ds}{c}$$

Try first:

$$t_{12} = \int_1^2 \frac{ds}{c} = \int_{\theta_0}^{\pi/2} \frac{Rd\theta}{c}$$

Using

$$R = \frac{c}{b\sin\theta}$$

gives

$$t_{12} = \frac{1}{b}\int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin\theta}$$

so that

$$t_{12} = \frac{1}{b}\left[\ln\tan\frac{\theta}{2}\right]_{\theta_0}^{\pi/2} = -\frac{1}{b}\ln\tan\frac{\theta_0}{2}$$

Noting that $t_{13} = 2t_{12}$ gives

$$t_{13} = -\frac{2}{b}\ln\tan\frac{\theta_0}{2} \quad .$$

For the specified b, this gives a transit time for the smallest value of launch angle cited in the answer to part (d), of

$$t_{13} = 6.6546 \text{ s}$$

The axial ray will have travel time given by

$$t = \frac{X}{c_0}$$

For the conditions given,

$$t_{13} = 6.6666 \text{ s}$$

thus this axial ray travels slower than the example cited for $n = 1$, thus the $n = 1$ ray will arrive first.