Solutions to Theoretical Question 1

Gravitational Red Shift and the Measurement of Stellar Mass

(a)

If a photon has an effective inertial mass m determined by its energy then $mc^2 = hf$ or $m = \frac{hf}{c^2}$. Now, assume that gravitational mass = inertial mass, and consider a photon of energy hf (mass $m = \frac{hf}{c^2}$) emitted upwards at a distance r from the centre of the star. It will lose energy on escape from the gravitational field of the star.

Apply the principle of conservation of energy:

Change in photon energy $(hf_i - hf_f)$ = change in gravitational energy, where subscript $i \rightarrow$ initial state and subscript $f \rightarrow$ final state.

$$\begin{aligned}
hf_i - hf_f &= -\frac{GMm_f}{\infty} - \left[-\frac{GMm_i}{r}\right] \\
hf_f &= hf_i - \frac{GMm_i}{r} \\
hf_f &= hf_i - \frac{GM\frac{hf_i}{c^2}}{r} \\
hf_f &= hf_i\left[1 - \frac{GM}{rc^2}\right] \\
\frac{f_f}{f_i} &= \left[1 - \frac{GM}{rc^2}\right] \\
\frac{\Delta f}{f} &= \frac{f_f - f_i}{f_i} = -\frac{GM}{rc^2}
\end{aligned}$$

The negative sign shows red-shift, i.e. a decrease in f, and an increase in wavelength.
Thus, for a photon emitted from the surface of a star of radius R, we have

$$\boxed{\frac{\Delta f}{f} = \frac{GM}{Rc^2}}$$

Since the change in photon energy is small, $(\delta f \ll f)$,

$$m_f \simeq m_i = \frac{hf_i}{c^2} \ .$$

(b)

The change in photon energy in ascending from r_i to r_f is given by

$$\begin{aligned}
hf_i - hf_f &= -\frac{GMm_f}{r_f} + \frac{GMm_i}{r_i} \\
&\simeq \frac{GMhf_i}{c^2}\left[\frac{1}{r_i} - \frac{1}{r_f}\right] \\
\therefore \frac{f_f}{f_i} &= 1 - \frac{GM}{c^2}\left[\frac{1}{r_i} - \frac{1}{r_f}\right]
\end{aligned}$$

In the experiment, R is the radius of the star, d is the distance from the surface of the star to the spacecraft and the above equation becomes:

$$\frac{f_f}{f_i} = 1 - \frac{GM}{c^2}\left[\frac{1}{R} - \frac{1}{R+d}\right] \tag{1}$$

The frequency of the photon must be doppler shifted back from f_f to f_i in order to cause resonance excitation of the He^+ ions in the spacecraft.
Thus apply the relativistic Doppler principle to obtain:

$$\frac{f'}{f_f} = \sqrt{\frac{1+\beta}{1-\beta}}$$

where f' is the frequency as received by He^+ ions in the spacecraft, and $\beta = v/c$.
That is, the gravitationally reduced frequency f_f has been increased to f' because of the velocity of the ions on the spacecraft towards the star. Since $\beta \ll 1$,

$$\frac{f_f}{f'} = (1-\beta)^{\frac{1}{2}}(1+\beta)^{-\frac{1}{2}} \simeq 1-\beta$$

Alternatively, since $\beta \ll 1$, use the classical Doppler effect directly.
Thus

$$f' = \frac{f_f}{1-\beta}$$

or

$$\frac{f_f}{f'} = 1-\beta$$

Since f' must be equal to f_i for resonance absorption, we have

$$\frac{f_f}{f_i} = 1-\beta \tag{2}$$

Substitution of 2 into 1 gives

$$\beta = \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+d}\right) \tag{3}$$

Given the experimental data, we look for an effective graphical solution. That is, we require a linear equation linking the experimental data in β and d.
Rewrite equation 3:

$$\beta = \frac{GM}{c^2}\left[\frac{\cancel{R}+d-\cancel{R}}{(R+d)R}\right]$$

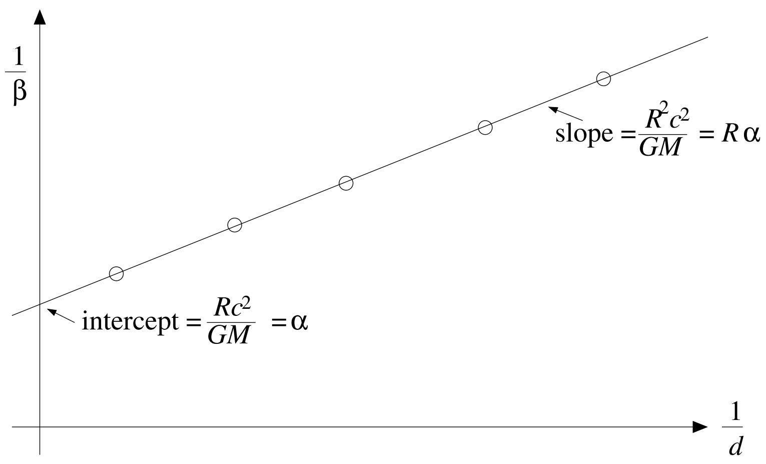
Inverting the equation gives:

$$\frac{1}{\beta} = \left(\frac{Rc^2}{GM}\right)\left[\frac{R}{d}+1\right]$$

or

$$\boxed{\frac{1}{\beta} = \left(\frac{R^2c^2}{GM}\right)\frac{1}{d} + \frac{Rc^2}{GM}}$$

Graph of $\frac{1}{\beta}$ vs. $\frac{1}{d}$



$$\text{The slope is } \left(\frac{Rc^2}{GM}\right)R = \alpha R \tag{A}$$

$$\text{The } \frac{1}{\beta}\text{-intercept is } \left(\frac{Rc^2}{GM}\right) = \alpha \tag{B}$$

$$\text{and the } \frac{1}{d}\text{-intercept is } -\frac{1}{R} \tag{C}$$

R and M can be conveniently determined from (A) and (B). Equation (C) is redundant. However, it may be used as an (inaccurate) check if needed.
From the given data:

$$R = 1.11 \times 10^8 \text{ m}$$
$$M = 5.2 \times 10^30 \text{ kg}$$

$$\text{From the graph, the slope} \quad \alpha R = 3.2 \times 10^{12} \text{ m} \qquad \text{(A)}$$
$$\text{The } \frac{1}{\beta}\text{-intercept} \quad \alpha = \frac{Rc^2}{GM} = 0.29 \times 10^5 \qquad \text{(B)}$$

Dividing (A) by (B)

$$R = \frac{3.2 \times 10^{12} \text{ m}}{0.29 \times 10^5} \simeq \boxed{1.104 \times 10^8 \text{ m}}$$

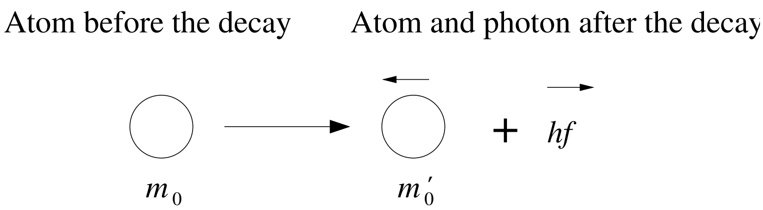
Substituting this value of R back into (B) gives:

$$M = \frac{Rc^2}{g\alpha} = \frac{(1.104 \times 10^8) \times (3.0 \times 10^8)^2}{(6.7 \times 10^{-11}) \times (0.29 \times^1 0^5)}$$

or $M = 5.11 \times 10^{30}$ kg

(c)

(i)



For the photon, photon momentum is $p = \dfrac{hf}{c}$ and photon energy is $E = hf$.
Use the mass-energy equivalence, $E = mc^2$, to relate the internal energy change of the atom to the rest-mass change. Thus:

$$\Delta E = (m_0 = m_0')\, c^2 \qquad (1)$$

In the laboratory frame of reference the energy before emission is

$$E = m_0 c^2 \qquad (2)$$

Recalling the relativistic relation

$$E^2 = p^2c^2 + m_0^2c^4$$

The energy after emission of a photon is

$$E = \sqrt{p^2c^2 + m_0'^2c^4} + hf \qquad (3)$$

where also $p = hf/c$ by conservation of momentum.
Conservation of energy requires that (2) = (3), so that:

$$\left(m_0c^2 - hf\right)^2 = (hf)^2 + m_0^2c^4$$
$$\left(m_0c^2\right)^2 - 2hfm_0c^2 = m_0^2c^4$$

Carrying out the algebra and using equation (1):

$$\begin{aligned} hf(2m_0c^2) &= (m_0^2 - m_0'^2)c^4 \\ &= (m_0 - m_0')c^2(m_0 + m_0')c^2 \\ &= \Delta E[2m_0 - (m_0 - m_0')]c^2 \\ &= \Delta E[2m_0c^2 - \Delta E] \end{aligned}$$

$$\boxed{hf = \Delta E \left[1 - \frac{\Delta E}{2m_0c^2}\right]}$$

(ii)

For the emitted photon,

$$hf = \Delta E \left[1 - \frac{\Delta E}{2m_0c^2}\right] \quad .$$

If relativistic effects are ignored, then

$$hf_0 = \Delta E \quad .$$

Hence the relativistic frequency shift $\frac{\Delta f}{f_0}$ is given by

$$\boxed{\frac{\Delta f}{f_0} = \frac{\Delta E}{2m_0c^2}}$$

For He^+ transition ($n = 2 \to 1$), applying Bohr theory to the hydrogen-like helium ion gives:

$$\Delta E = 13.6 \times 2^2 \times \left[\frac{1}{1^2} - \frac{1}{2^2}\right] = 40.8 \text{ ev}$$

Also, $m_0c^2 = 3.752 \times 10^6$ eV. Therefore the frequency shift due to the recoil gives

$$\boxed{\frac{\Delta f}{f_0} \simeq 5.44 \times 10^{-12}}$$

This is very small compared to the gravitational red-shift of $\frac{\Delta f}{f} \sim 10^{-5}$, and may be ignored in the gravitational red-shift experiment.