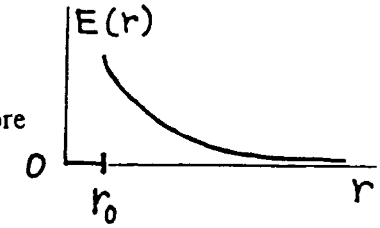
Theoretical Problem 3 -- Solution

1. By symmetry, the electric field will point radially away from the wire, and its magnitude will depend only on the radius r (in cylindrical coordinates). Place an imaginary cylinder around the wire and use Gauss's law:

$$2\pi r E(r) = \frac{q_{\text{linear}}}{\varepsilon_0}$$



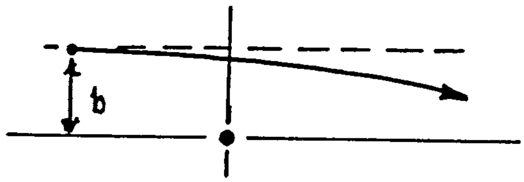
for a cylinder of radius r and unit length, provided $r \geq r_0$. Therefore

$$E(r) = \frac{q_{\text{linear}}}{2\pi r \varepsilon_0} = \frac{0.791}{r} N/C \quad \text{provided} \quad r \geq r_0.$$

When $r < r_0$, the electric field is zero (because copper is a good conductor), that is, the electric field is zero inside the wire.

2. The problem stated that the angular deflection is small. Estimate the deflection angle θ_{final} by forming a quotient: the momentum acquired transverse to the initial velocity divided by the initial momentum:

$$\theta_{\text{final}} \cong \frac{|\Delta \mathbf{p}_\perp|}{m v_0}.$$



A first estimate of the transverse momentum can be made as follows:

The transverse force (where it is significant) is of order $\frac{e q_{\text{linear}}}{2\pi\varepsilon_0 b}$.

The (significant) transverse force operates for a time such that the electron goes a distance of order $2b$, and hence that transverse force operates for a time of order $2b/v_0$.

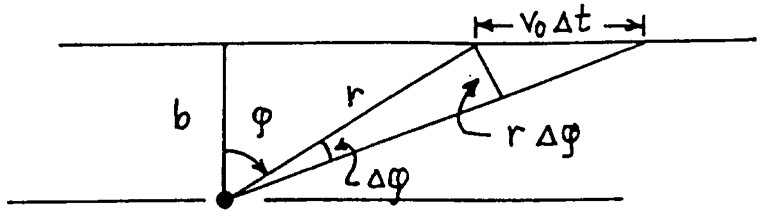
The product of force and operating time gives an estimate the transverse momentum:

$$|\Delta \mathbf{p}_\perp| \cong \frac{e q_{\text{linear}} 2b}{2\pi\varepsilon_0 b v_0} = \frac{e q_{\text{linear}}}{\pi\varepsilon_0 v_0},$$

and so $\theta_{\text{final}} \cong \frac{e q_{\text{linear}}}{\pi\varepsilon_0 m v_0^2} = \frac{q_{\text{linear}}}{\pi\varepsilon_0 2V_0} = 3.96 \times 10^{-5}$ radians

after one uses energy conservation to say $\frac{1}{2} m v_0^2 = eV_0$. Note that the deflection is extremely small and that the deflection is independent of the impact parameter b. Because the force between the positively charged wire and the electron is attractive, the deflection will bend the trajectory toward the wire--though only ever so slightly.

A more accurate estimate can be made by setting up an elementary integration for $|\Delta \mathbf{p}_\perp|$, as follows. For the sake of the integration, approximate the actual trajectory by a straight line that passes the wire at distance b, as shown in the sketch.



$$|\mathbf{F}_\perp| = \frac{e q_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi \qquad v_0\,\Delta t\cos\varphi = r\,\Delta\varphi \quad \text{and so} \quad \Delta t = \frac{r\,\Delta\varphi}{v_0\cos\varphi}$$

$$|\mathbf{F}_\perp|\,\Delta t = \frac{e q_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi\frac{r\,\Delta\varphi}{v_0\cos\varphi} = \frac{e q_{\text{linear}}}{2\pi\varepsilon_0 v_0}\Delta\varphi.$$

Adding up the increments in $\Delta\varphi$ over the range $-\pi/2$ to $\pi/2$ yields $|\Delta \mathbf{p}_\perp| = \frac{e q_{\text{linear}}}{2\varepsilon_0 v_0}$.

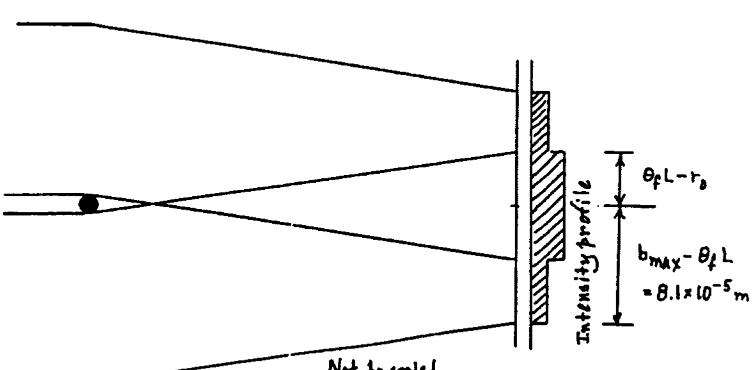
The better estimate differs from the first estimate by merely the factor $\pi/2$. The better estimate yields

$$\theta_{\text{final}} \cong \frac{e q_{\text{linear}}}{2\varepsilon_0 m v_0^2} = \frac{q_{\text{linear}}}{2\varepsilon_0 2V_0} = 6.21\times10^{-5}\ \text{radians}.$$

3. Most of the bending of the trajectory occurs within a distance from the wire of order b. On the scale of L, order b is very small indeed. Therefore we may approximate the trajectory by two straight lines with a kink near the wire. Thus, at the viewing surface, the transverse displacement of each trajectory is

$$\begin{pmatrix}\text{transverse}\\ \text{displacement}\end{pmatrix} = \theta_{\text{final}} L = 6.21\times10^{-5}\times0.3 = 1.86\times10^{-5}\ \text{meter} \cong 19\,r_0 >> r_0.$$

Thus the portions of the beam that pass on opposite sides of the wire have a region of overlap, as shown in the sketch.



The full width of the overlap region is

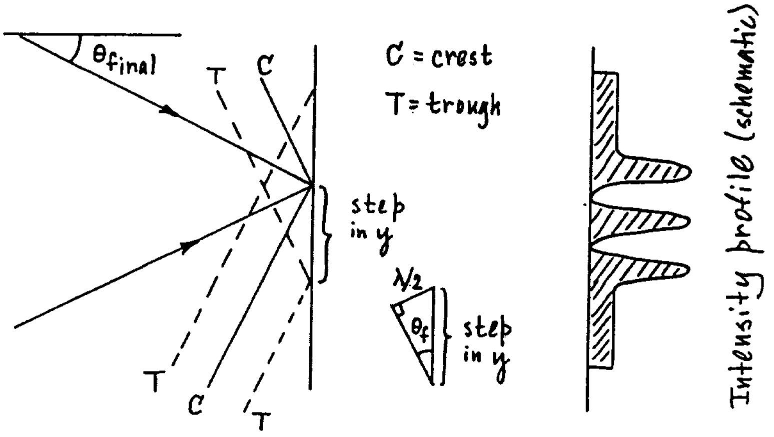
$$\begin{pmatrix} \text{full} \\ \text{width} \end{pmatrix} = 2 \times (\theta_{\text{final}} L - r_0) \cong 36\, r_0 = 36 \times 10^{-6} \quad \text{meter}.$$

The density of impacts is constant within each region and doubled in the overlap region.

4. Associated with the electron beam is a quantum wave pattern whose de Broglie wavelength is

$$\lambda = \frac{h}{m v_0} = \frac{h}{\sqrt{2meV_0}} = 8.68 \times 10^{-12} \quad \text{meter}.$$

The de Broglie wavelength is so much smaller than the beam width $2b_{max}$ that one may ignore "single slit diffraction" effects. Rather, to the right of the wire, two plane waves that travel at a fixed angle relative to each other (an angle $2\theta_{final}$) overlap and interfere. In the region where, classically, the two halves of the original beam overlap, there will be interference maxima and minima.



Reference to the sketch indicates that

$$\begin{pmatrix} \text{Interval between} \\ \text{adjacent constructive} \\ \text{interference locations} \end{pmatrix} = \begin{pmatrix} \text{step} \\ \text{in } y \end{pmatrix} = \frac{\lambda/2}{\sin\theta_{\text{final}}} \cong \frac{\lambda/2}{\theta_{\text{final}}} \cong \frac{\frac{1}{2} \times 8.68 \times 10^{-12}}{6.21 \times 10^{-5}} = 7.00 \times 10^{-8} \quad \text{meter}.$$

Because the region of overlap has a full width of $\cong 36 \times 10^{-6}$ meter, there will be roughly 500 interference maxima. Note that the interval between adjacent maxima does *not* depend on either b or b_{max} (unlike the situation with ordinary "double slit interference").

Historical note. This problem is based on the now-classic experiment by G. Mollenstedt and H. Duker, "Observation and Measurement of Biprism Interference with Electron Waves," Zeitschrift für Physik, 145, pp. 377-397 (1956).

Theoretical Problem 3: Grading Scheme

Part 1. 1 point.

E(r) correct outside of wire: 1 point.

E(r) inside wire: ignore in the grading. (Some students may ignore the interior because there is no field there.)

Part 2. 5 points, distributed as follows:

θ_{final} independent of b: 1 pt.

$\theta_{\text{final}} \propto \frac{e q_{\text{linear}}}{\varepsilon_0 m v_0^2}$ or $\frac{q_{\text{linear}}}{\varepsilon_0 V_0}$ or equivalent: + 1 pt.

Numerical coefficient correct to within a factor of 4: + 2 pts.

Numerical coefficient correct to within 20 %: + 1 pt.

Part 3: 1.5 points:

Overlap region exists: 0.5 pt.

Constant densities of impacts within each region: + 0.25 pt.

Correct ratio of intensities: + 0.25 pt.

Full width of pattern correct, given student's value for θ_{final}: + 0.25 pt.

Width of overlap region correct, given student's value for θ_{final}: + 0.25 pt.

Part 4: 2.5 points:

Recognizes that "two wave" interference occurs: 0.5 pt.

Correct de Broglie wavelength : 0.5 pt.

Correct separation of maxima: + 1.5 pts.

[If separation of maxima is wrong by merely a factor of 2, then partial credit: +1 pt.]

Maxima in intensity = 4 times single-wave intensity: ignore in grading.