Theoretical Problem 2 -- Solution

1. This is a simple problem in geometry and Snell's Law

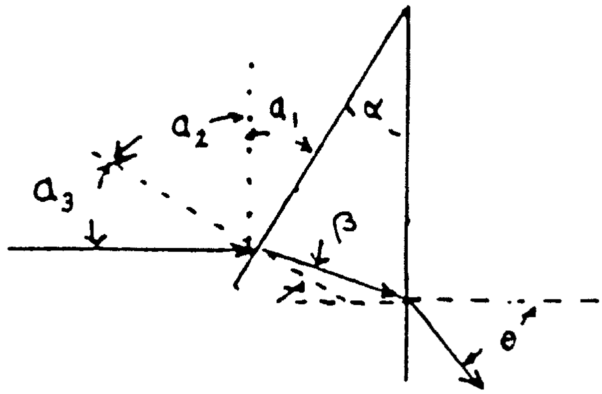

Figure 1: Refraction through a wedge.

The angle of incidence $a_3 = \alpha$ because $a_1 = \alpha$ and $a_1 + a_2 = a_2 + a_3 = 90°$. The angle β is found from Snell's law $\sin\alpha = n \sin\beta$. The angle of incidence on the base is

$$\frac{\pi}{2} - (\pi - \alpha - (\frac{\pi}{2} - \beta)) = \alpha - \beta$$

from which it follows that

$$\sin\theta = n \sin(\alpha - \beta)$$

implying that

$$\theta = \sin^{-1}\left[n \sin\left(\alpha - \sin^{-1}\left(\frac{\sin\alpha}{n}\right)\right)\right]$$

2. The force on the prism is equal and opposite to the rate of change of momentum of the laser light passing through it. To analyze this, consider the momentum changes of the laser light incident on the upper half of the prism.

   Think of the laser beam as delivering to the upper half of the prism r_u photons per second parallel to the x axis. If the energy of a photon is E, then its momentum is $\vec{p}_i = \frac{E}{c}\,\hat{\imath}$, and a photon leaving the prism at an angle θ to the x axis will differ in momentum from the incident photon by

$$\delta\vec{p} = \frac{E}{c}(\cos\theta - 1)\,\hat{\imath} - \frac{E}{c}\sin\theta\,\hat{\jmath}.$$

   The rate of charge of momentum of these photons will then be

$$\vec{F}_{up} = r_u \delta\vec{p} = \frac{r_u E}{c}\left[(\cos\theta - 1)\,\hat{\imath} - \sin\theta\ \hat{\jmath}.\right]$$

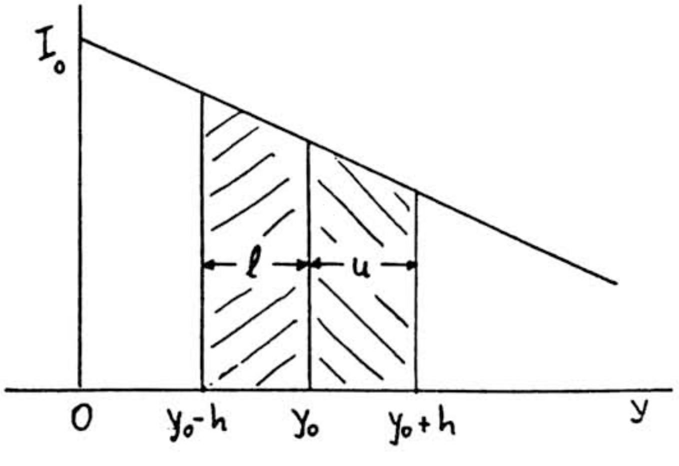

Figure 2: $\bar{I}_u$ and $\bar{I}_\ell$ when $y_0 \geq h$

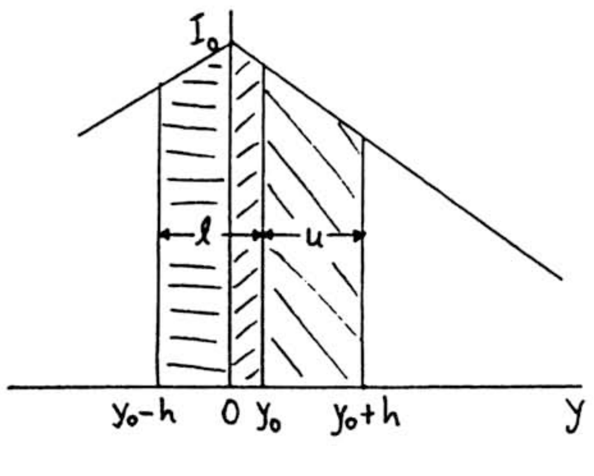

Figure 3: $\bar{I}_u$ and $\bar{I}_\ell$ when $0 < y_0 < h$

The quantity $r_u E$ is the power P_u delivered to the upper face, and the recoil force $\vec{F}_u$ produced by light refracting through the upper half of the prism will be

$$\vec{F}_u = \frac{P_u}{c}\left[(1-\cos\theta)\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right].$$

A similar argument gives the force on the lower half as

$$\vec{F}_\ell = \frac{P_\ell}{c}\left[(1-\cos\theta)\,\hat{\imath} - \sin\theta\,\hat{\jmath}\right].$$

From these two results we see that the net force on the prism will be

$$\vec{F} = \frac{1}{c}\left[(P_u + P_\ell)(1-\cos\theta)\right]\,\hat{\imath} + \frac{1}{c}\left[(P_u - P_\ell)\,\sin\theta\right]\,\hat{\jmath}.$$

The angle θ can be expressed in terms of α (see answer to part 1).

To find the values of P_u and P_ℓ calculate the average intensities, $\bar{I}_u$ and $\bar{I}_\ell$, incident on each half of the prism and multiply by hw, the area of each half of the prism projected perpendicular to the laser beam. Because the intensity distribution $I(y)$ is a linear function of y, the average intensities are easily determined.

The problem states that

$$\begin{aligned} I(y) &= I_0\left(1-\frac{y}{4h}\right) && \text{for } 0 < y < +4h \\ &= I_0\left(1+\frac{y}{4h}\right) && \text{for } -4h < y < 0. \end{aligned}$$

Now suppose that the prism is lifted a distance y_0 from the x axis ($y_0 > 0$). There are two distinct cases:

(a) When $h \leq y_0 \leq 3h$, the whole prism is entirely in the upper half of the beam. As Fig. 2 shows, for this case the average is equal to the intensity at the center of each face which is at $y_0 + h/2$ for the upper face and at $y_0 - h/2$ for the lower one. This gives

$$\bar{I}_u = I_0\left(1-\frac{y_0+h/2}{4h}\right) = I_0\left(\frac{7}{8}-\frac{y_0}{4h}\right)$$
$$\bar{I}_\ell = I_0\left(1-\frac{y_0-h/2}{4h}\right) = I_0\left(\frac{9}{8}-\frac{y_0}{4h}\right)$$

From these it follows that

$$\begin{aligned} F_x &= \frac{2hwI_0}{c}\left(1-\frac{y_0}{4h}\right)(1-\cos\theta) \\ F_y &= -\frac{hwI_0}{4c}\sin\theta. \end{aligned}$$

(b) When $0 < y_0 < h$, the lower half of the prism is partly in the lower half of the laser beam as shown in Fig. 3. Then the part of the lower half of the prism between 0 and y_0 has a fraction y_0/h of the area of the lower half of the prism and sees an average intensity

$$\bar{I}_{\ell_1} = I(y_0/2) = I_0 \left(1 - \frac{y_0}{8h}\right).$$

The part between 0 and $y_0 - h$ has a fraction $1 - y_0/h$ of the area and sees an average intensity of

$$\bar{I}_{\ell_2} = I(\frac{h - y_0}{2}) = I_0 \left(\frac{7}{8} + \frac{y_0}{8h}\right).$$

Putting these together we get

$$\begin{aligned} P_\ell &= hw\,\frac{y_0}{h}\bar{I}_{\ell_1} + hw\,(1 - \frac{y_0}{h})\bar{I}_{\ell_2} \\ &= hwI_0 \left(\frac{7}{8} + \frac{y_0}{4h} - \frac{y_0^2}{4h^2}\right). \end{aligned}$$

The average intensity on the upper face has the same functional dependence on y_0 as in the first case. Therefore, $P_u = hwI_0 \left(\frac{7}{8} - \frac{y_0}{4h}\right)$ as before.

Putting these together gives

$$\begin{aligned} P_u + P_\ell &= hwI_0 \left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right) \\ P_u - P_\ell &= -hwI_0\frac{y_0}{2h} \left(1 - \frac{y_0}{2h}\right) \end{aligned}$$

from which it follows that

$$\begin{aligned} F_x &= \frac{hwI_0}{c} \left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right)(1 - \cos\theta) \\ F_y &= -\frac{hwI_0}{c}\frac{y_0}{2h} \left(1 - \frac{y_0}{2h}\right)\sin\theta. \end{aligned}$$

Because the intensity distribution is symmetric about the axis of the laser beam, the solutions for $y_0 < 0$ will mirror the solutions for $y_0 > 0$. Graphs of the F_x and F_y as functions of y_0 are shown in Fig. 4.

3. Both the equation and the graph of F_y show that to have $F_y > 0$ and opposite the force of gravity, y_0 must be < 0. Then to find the force necessary to support the prism against gravity, find the prism's mass, and equate the expression for the vertical component of force from the laser beam to the weight of the prism, and find I_0 for the parameters given. Use that result to find the total power in the laser beam. This

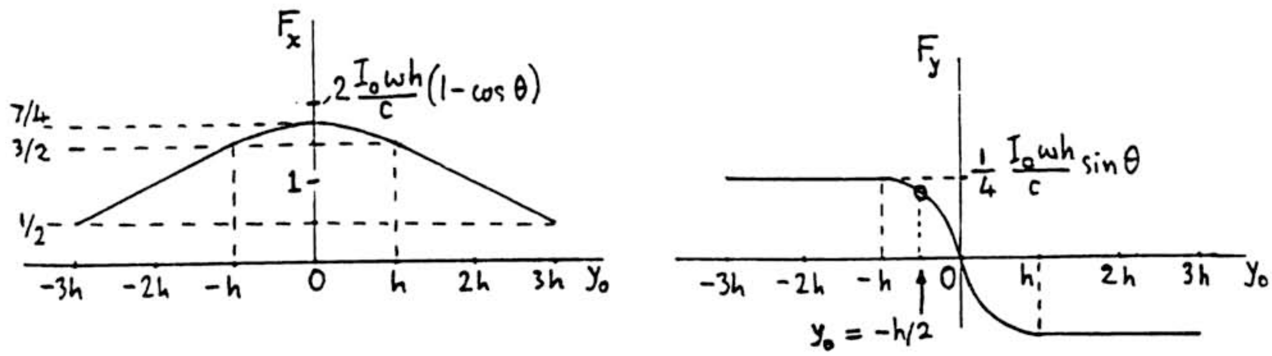

Figure 4: (a) F_x vs y_0; (b) F_y vs y_0

can be done by finding the average value $\bar{I}$ over the specified cross sectional area of the laser beam.

To find the mass of the prism first find its volume $= \tan\alpha\ h^2 w$ then

$$\begin{aligned} m &= \frac{1}{\sqrt{3}} \times (10^{-3})^2 \times .1 \times 2.5 \\ &= 1.44 \times 10^{-7}\ \text{g} \\ &= 1.44 \times 10^{-10}\ \text{kg}; \\ mg &= 1.42 \times 10^{-9}\ \text{N} \end{aligned}$$

The solution to (2) assumed a displacement in the $y > 0$ direction, but the problem is symmetric so we can use that solution. We want the value of I_0 that satisfies

$$\frac{I_0 h w}{c}\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right)\sin\theta = mg = 1.42 \times 10^{-9}$$

when

$$\begin{aligned} \theta &= 15.9^\circ \\ y_0 &= \frac{h}{2} \\ h &= 10 \times 10^{-6}\ \text{m} \\ w &= 10^{-3}\ \text{m} \end{aligned}$$

$$I_0 = \frac{3 \times 10^8 \times 1.42 \times 10^{-9}}{10^{-5} \times 10^{-3} \times .274 \times \frac{3}{16}} = 8.30 \times 10^8\ \text{W/m}^2$$

since the power P is given by $P = \bar{I} \times$ area of laser beam where $\bar{I} = \frac{I_0}{2}$. This yields

$$P = \frac{1}{2} \times 8.30 \times 10^8 \times 10^{-3} \times 80 \times 10^{-6} = 33.2\ \text{W}.$$

4. A displacement of $h/20$ corresponds to $y_0/h = .05 << 1$ so that the vertical force component is well approximated by

$$F_y = -\frac{I_0 w \sin\theta}{2c} y.$$

This is the equation of a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{I_0 w \sin\theta}{2mc}} = \sqrt{\frac{I_0 \sin\theta}{2c\rho h^2 \tan\alpha}}.$$

Putting numbers into this gives

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2 \times 3 \times 10^8 \times 2.5 \times 10^3 \times 10^{-10} \times 1/\sqrt{3}}{10^8 \times .274}} = 11.2\times10^{-3} \text{ s}.$$

Theoretical Problem 2: Grading Scheme

Part 1.	1.5 points	
Part 2.	5 points	(2 points for obtaining expression for net force in terms of θ and powers P_u, P_l incident on upper and lower prism faces ; 1 point for finding F_x and F_y explicitly in terms of I_0, y_0 and θ for $h \le y_0 \le 3h$; 1 point for finding F_x and F_y explicitly in terms of I_0, y_0 and θ for $0 \le y_0 \le h$; 1 point for drawing appropriate graphs)
Part 3.	1.5 points	
Part 4.	2 points	