SOLUTION : PROBLEM 3

1. Over the whole surface of the sun, the emitted energy is $4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4$. All this energy passes through a spherical shell at earth's distance R, where the intensity now is $4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4 / 4\pi R^2$.
   The satellite is a circular object absorbing
$$\pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2}$$
   but a spherical object emitting $4\pi r_{sat}^2 \cdot \sigma T_{sat}^4$.
   Equating this absorption and emission we get $T_{sat} = T_{Sun}\sqrt{\frac{R_{Sun}}{2R}}$ giving $T_{sat} = 289\ \text{K} = 16\ °\text{C}$.

2. We have to calculate what part of the absorbed power comes from the part of the spectrum below 1200 K.
$$\eta_{cutoff} = \frac{1200\ \text{K}}{6000\ \text{K}} = 0.2 \ll 1$$
   This fraction of power is
$$\delta = \int_0^{\eta_{cutoff}} \frac{\eta^3\, d\eta}{e^\eta - 1} \Big/ \int_0^{\eta_\infty} \frac{\eta^3\, d\eta}{e^\eta - 1}$$
$$\approx \int_0^{\eta_{cutoff}} \eta^2\, d\eta \Big/ \frac{\pi^4}{15} = \frac{\eta_{cutoff}^3}{3} \Big/ \frac{\pi^4}{15} = 4.1 \cdot 10^{-4}$$
   Now, the satellite is cold with respect to 1200 K so we ignore that a small part of the satellite blackbody emission will be retained. The energy balance is now
$$4\pi r_{sat}^2 \cdot \sigma T_{sat}^4 = \delta \cdot \pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2}$$
   by which the new satellite temperature is the previous corrected by a factor $\delta^{1/4}$
$$T_{sat} = (4.1 \cdot 10^{-4})^{1/4} \cdot 289\ \text{K} = 41\ \text{K}$$

3. The whole absorbed energy is
$$\delta \cdot \pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2} = 0.5\ \text{W}$$
   which is small (ignorable) compared to $P_{internal} = 1$ kW. Thus the energy balance becomes
$$P_{internal} = 4\pi r_{sat}^2 \cdot \sigma T_{sat}^4$$

giving $T_{sat} = 274$ K ($\eta = 4.38$). Note: strictly speaking, this is not accurate, because for a blackbody radiation of 274 K, some 33 % of the emitted power lies above the 1200 K cutoff ! This means that the satellite has to be hotter, to emit all of the 1 kW in frequencies below the cutoff. The resulting integral equation is

$$(\frac{\eta}{4.38})^4 = \int_0^\eta \frac{\eta^3\,d\eta}{e^\eta - 1}/\frac{\pi^4}{15}$$

which can be solved numerically by iteration. The true solution is $\eta = 3.80$ corresponding to a temperature of 316 K.

4. The paint cannot exist, because it would violate the second law of thermodynamics. The physics textbook explanation is the principle of detailed balance, which means that for equilibrium to exist, the emission and absorption in a given frequency interval must match exactly. This should not be confused with the fact that reflection and absorption can be quite different. If the manufacturer's paint existed, one could create a temperature difference between two bodies in a closed system, and hence a perpetum mobile.

5. The coating should be transparent for high frequencies (in the range of the peak or tail of the sun's radiation), but reflective and hence insulating at low frequencies (the satellite temperature).