Solution

As the system is isolated, its total energy, i.e. the sum of the kinetic and potential energies, is conserved. The total potential energy of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 in the inertial system (i.e. when there are no inertial forces) is equal to the sum of the gravitational potential energies of all the pairs of points (P_1,P_2), (P_2,P_3) and (P_1,P_3). It depends only on the distances a_{12}, a_{23} and a_{23} which are constant in time. Thus, the total potential energy of the system is constant. As a consequence the kinetic energy of the system is constant too. The moment of inertia of the system with respect to the axis σ depends only on the distances from the points P_1, P_2 and P_3 to the axis σ which, for fixed a_{12}, a_{23} and a_{23} do not depend on time. This means that the moment of inertia I is constant. Therefore, the angular velocity of the system must also be constant:

$$\omega = \text{const.} \tag{1}$$

This is the first condition we had to find. The other conditions will be determined by using three methods described below. However, prior to performing calculations, it is desirable to specify a convenient coordinates system in which the calculations are expected to be simple.

Let the positions of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 be given by the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$. For simplicity we assume that the origin of the coordinate system is localized at the center of mass of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 and that all the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$ are in the same coordinate plane, e.g. in the plane (*x,y*). Then the axis σ is the axis z.

In this coordinate system, according to the definition of the center of mass, we have:

$$m_1\mathbf{r_1} + m_2\mathbf{r_2} + m_3\mathbf{r_2} = 0 \tag{2}$$

Now we will find the second condition by using several methods.

FIRST METHOD

Consider the point P_1 with the mass m_1. The points P_2 and P_3 act on it with the forces:

$$\mathbf{F}_{21} = G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1), \tag{3}$$

$$\mathbf{F}_{31} = G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1). \tag{4}$$

where *G* denotes the gravitational constant.

In the inertial frame the sum of these forces is the centripetal force

$$\mathbf{F}_{r1} = -m_1\omega^2\mathbf{r}_1,$$

which causes the movement of the point P_1 along a circle with the angular velocity ω. (The moment of this force with respect to the axis σ is equal to zero.) Thus, we have:

$$\mathbf{F}_{21} + \mathbf{F}_{31} = \mathbf{F}_{r1}. \tag{5}$$

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the sum of the forces (3), (4) and the centrifugal force

$$\mathbf{F'}_{r1} = m_1\omega^2\mathbf{r}_1$$

should be equal to zero:

$$\mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}'_{r1} = 0. \tag{6}$$

(The moment of this sum with respect to any axis equals to zero.)

The conditions (5) and (6) are equivalent. They give the same vector equality:

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1\omega^2\mathbf{r}_1 = 0, \tag{7'}$$

$$G\frac{m_1}{a_{12}^3} m_2\mathbf{r}_2 + G\frac{m_1}{a_{13}^3} m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0 \tag{7''}$$

From the formula (2), we get:

$$m_2\mathbf{r}_2 = -m_1\mathbf{r}_1 - m_3\mathbf{r}_3 \tag{8}$$

Using this relation, we write the formula (7) in the following form:

$$G\frac{m_1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3) + G\frac{m_1}{a_{13}^3} m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0,$$

i.e.

$$\mathbf{r}_1 m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) + \mathbf{r}_3\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-col1inear. Therefore, the coefficients in the last formula must be equal to zero:

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0,$$

$$m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) = 0.$$

The first equality leads to:

$$\frac{1}{a_{13}^3} = \frac{1}{a_{12}^3}$$

and hence,

$$a_{13} = a_{12}.$$

Let $a_{13} = a_{12} = a$. Then the second equality gives:

$$\omega^2 a^3 = GM \tag{9}$$

where

$$M = m_1 + m_2 + m_3 \tag{10}$$

denotes the total mass of the system.

In the same way, for the points P_2 and P_3, one gets the relations:

a) the point P_2:

$$a_{23} = a_{12}; \qquad \omega^2 a^3 = GM$$

b) the point P_3:

$$a_{13} = a_{23}; \qquad \omega^2 a^3 = GM$$

Summarizing, the system can rotate as a rigid body if all the distances between the masses are equal:

$$a_{12} = a_{23} = a_{13} = a, \tag{11}$$

the angular velocity ω is constant and the relation (9) holds.

SECOND METHOD

At the beginning we find the moment of inertia I of the system with respect to the axis σ. Using the relation (2), we can write:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = m_1^2\mathbf{r}_1^2 + m_2^2\mathbf{r}_2^2 + m_3^2\mathbf{r}_3^2 + 2m_1m_2\mathbf{r}_1\mathbf{r}_2 + 2m_1m_3\mathbf{r}_1\mathbf{r}_3 + 2m_3m_2\mathbf{r}_3\mathbf{r}_2.$$

Of course,

$$\mathbf{r}_i^2 = r_i^2 \qquad\qquad i = 1, 2, 3$$

The quantities $2\mathbf{r}_i\mathbf{r}_j$ $(i, j = 1, 2, 3)$ can be determined from the following obvious relation:

$$a_{ij}^2 = \left|\mathbf{r}_i - \mathbf{r}_j\right|^2 = (\mathbf{r}_i - \mathbf{r}_j)^2 = \mathbf{r}_i^2 + \mathbf{r}_j^2 - 2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - 2\mathbf{r}_i\mathbf{r}_j.$$

We get:

$$2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - a_{ij}^2.$$

With help of this relation, after simple transformations, we obtain:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = (m_1 + m_2 + m_3)(m_1r_1^2 + m_2r_2^2 + m_3r_3^2) - \sum_{i<j} m_i m_j a_{ij}^2.$$

The moment of inertia I of the system with respect to the axis σ, *a*ccording to the definition of this quantity, is equal to

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2.$$

The last two formulae lead to the following expression:

$$I = \frac{1}{M}\sum_{i<j} m_i m_j a_{ij}^2$$

where M (the total mass of the system) is defined by the formula (10).

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the total potential energy V_{tot} is the sum of the gravitational potential energies

$$V_{ij} = -G\frac{m_i m_j}{a_{ij}}; \qquad\qquad i, j = 1, 2, 3;\ i < j$$

of all the masses and the potential energies

$$V_i = -\tfrac{1}{2}\omega^2 m_i r_i^2; \qquad\qquad i = 1, 2, 3$$

of the masses m_i (i = 1, 2, 3) in the field of the centrifugal force:

$$V_{tot} = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\sum_{i=1}^{3} m_i r_i^2 = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2 I = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\frac{1}{M}\sum_{i<j} m_i m_j a_{ij}^2 =$$

$$= -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right)$$

i.e.

$$V_{tot} = -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right).$$

A mechanical system is in equilibrium if its total potential energy has an extremum. In our case the total potential energy V_{tot} is a sum of three terms. Each of them is proportional to:

$$f(a) = \frac{\omega^2}{2M}a^2 + \frac{G}{a}.$$

The extrema of this function can be found by taking its derivative with respect to a and requiring this derivative to be zero. We get:

$$\frac{\omega^2}{M}a - \frac{G}{a^2} = 0.$$

It leads to:

$$\omega^2 a^3 = GM \quad \text{or} \quad \omega^2 a^3 = G(m_1 + m_2 + m_3).$$

We see that all the terms in V_{tot} have extrema at the same values of $a_{ij} = a$. (In addition, the values of a and ω should obey the relation written above.) It is easy to show that it is a maximum. Thus, the quantity V_{tot} has a maximum at $a_{ij} = a$.

This means that our three masses can remain in fixed distances only if these distances are equal to each other:

$$a_{12} = a_{23} = a_{13} = a$$

and if the relation

$$\omega^2 a^3 = GM ,$$

where M the total mass of the system, holds. We have obtained the conditions (9) and (11) again.

THIRD METHOD

Let us consider again the point P_1 with the mass m_1 and the forces $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ given by the formulae (3) and (4). It follows from the text of the problem that the total moment (with respect to any fixed point or with respect to the mass center) of the forces acting on the point P_1 must be equal to zero. Thus, we have:

$$\mathbf{F}_{21} \times \mathbf{r}_1 + \mathbf{F}_{31} \times \mathbf{r}_1 = 0$$

where the symbol $\times$ denotes the vector product. Therefore

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1)\times\mathbf{r}_1 + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1)\times\mathbf{r}_1 = 0.$$

But

$$\mathbf{r}_1 \times \mathbf{r}_1 = 0.$$

Thus:

$$\frac{m_2}{a_{12}^3}\mathbf{r}_2 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

Using the formula (8), the last relation can be written as follows:

$$\frac{1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3)\times\mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$-\frac{m_3}{a_{12}^3}\mathbf{r}_3 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-col1inear (and different from 0). Therefore

$$\mathbf{r}_3 \times \mathbf{r}_1 \neq 0$$

and

$$\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3} = 0,$$

hence,

$$a_{12} = a_{13}.$$

Similarly, one gets:

$$a_{12} = a_{23} \ \ (= a).$$

We have re-derived the condition (11).

Taking into account that all the distances a_{ij} have the same value a, from the equation (7) concerning the point P_1, using the relation (2) we obtain:

$$G\frac{m_1 m_2}{a^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1\omega^2\mathbf{r}_1 = 0,$$

$$-\left(G\frac{m_1}{a^3} + G\frac{m_2}{a^3} G\frac{m_3}{a^3}\right)m_1\mathbf{r}_1 + m_1\omega^2\mathbf{r}_1 = 0,$$

$$\frac{GM}{a^3} = \omega^2.$$

This is the condition (9). The same condition is got in result of similar calculations for the points P_2 and P_3.

The method described here does not differ essentially from the first method. In fact they are slight modifications of each other. However, it is interesting to notice how application of a proper mathematical language, e.g. the vector product, simplifies the calculations.

The relation (9) can be called a "generalized Kepler's law" as, in fact, it is very similar to the Kepler's law but with respect to the many-body system. As far as I know this generalized Kepler's law was presented for the first time right at the 20th IPhO.

Marking scheme

1. the proof that $\omega = \text{const}$	1 point
2. the conditions at the equilibrium (conditions for the forces and their moments or extremum of the total potential energy)	3 points
3. the proof of the relation $a_{ij} = a$	4 points
4. the proof of the relation $\omega^2 a^3 = GM$	2 points

Remarks and typical mistakes in the pupils' solutions

No type of error was observed as predominant in the pupils' solutions. Practically all the mistakes can be put down to the students' scant experience in calculations and general lack of skill. Several students misunderstood the text of the problem and attempted to prove that the three masses should be equal. Of course, this was impossible. Moreover, it was pointless, since the masses were given. Almost all the participants tried to solve the problem by analyzing equilibrium of forces and/or their moments. Only one student tried to solve the problem by looking for a minimum of the total potential energy (unfortunately, his solution was not fully correct). Several participants solved the problem using a convenient reference system: one mass in the origin and one mass on the x-axis. One of them received a special prize.