Solution

3.1

$$r_0^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$
$$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2$$

since

$$\Delta p_x \geq \frac{\hbar}{2 \cdot \Delta x} \qquad \Delta p_y \geq \frac{\hbar}{2 \cdot \Delta y} \qquad \Delta p_z \geq \frac{\hbar}{2 \cdot \Delta z}$$

gives

$$p_0^2 \geq \frac{\hbar^2}{4} \cdot \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right]$$

and

$$(\Delta x)^2 = (\Delta y)^2 = (\Delta z)^2 = \frac{r_0^2}{3}$$

thus $\boxed{p_0^2 \cdot r_0^2 \geq \frac{9}{4} \cdot \hbar^2}$

3.2

$|\vec{v}_e|$ ...... speed of the external electron before the capture

$|\vec{V}_i|$ .... speed of $A^{(Z-1)+}$ before capturing

$|\vec{V}_f|$ ...... speed of $A^{(Z-1)+}$ after capturing

$E_n = h.\nu$ ...... energy of the emitted photon

conservation of energy:

$$\frac{1}{2} \cdot m_e \cdot v_e^2 + \frac{1}{2} \cdot (M + m_e) \cdot V_i^2 + E\left[A^{(Z-1)+}\right] = \frac{1}{2} \cdot (M + 2 \cdot m_e) \cdot V_f^2 + E\left[A^{(Z-2)+}\right]$$

where $E[A^{(Z-1)+})$ and $E[A^{(Z-2)+}]$ denotes the energy of the electron in the outermost shell of ions $A^{(Z-1)+}$ and $A^{(Z-2)+}$ respectively.

conservation of momentum:

$$m_e \cdot \vec{v}_e + (M + m) \cdot \vec{V}_i = (M + 2 \cdot m_e) \cdot \vec{V}_f + \frac{h \cdot \nu}{c} \cdot \vec{1}$$

where $\vec{1}$ is the unit vector pointing in the direction of the motion of the emitted photon.

3.3

Determination of the energy of $A^{(Z-1)+}$:

potential energy $= -\dfrac{Z \cdot e^2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_0} = -\dfrac{Z \cdot q^2}{r_0}$

kinetic energy $= \dfrac{p^2}{2 \cdot m}$

If the motion of the electrons is confined within the x-y-plane, principles of uncertainty in 3.1 can be written as

$$r_0^2 = (\Delta x)^2 + (\Delta y)^2$$

$$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2$$

$$p_0^2 = \frac{\hbar^2}{4} \cdot \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right] = \frac{\hbar^2}{4} \cdot \left[\frac{2}{r_0^2} + \frac{2}{r_0^2} \right] = \frac{\hbar^2}{4} \cdot \frac{4}{r_0^2}$$

thus

$$p_0^2 \cdot r_0^2 = \hbar^2$$

$$E\left[A^{(Z-1)+}\right] = \frac{p_0^2}{2 \cdot m_e} - \frac{Z \cdot q^2}{r_0} = \frac{\hbar^2}{2 \cdot m_e \cdot r_e} - \frac{Z \cdot q^2}{r_0}$$

Energy minimum exists, when $\dfrac{dE}{dr_0} = 0$.

Hence

$$\frac{dE}{dr_0} = -\frac{\hbar^2}{m_e \cdot r_e^3} + \frac{Z \cdot q^2}{r_0^2} = 0$$

this gives $\dfrac{1}{r_0} = \dfrac{Z \cdot q^2 \cdot m_e}{\hbar^2}$

hence

$$E\left[A^{(Z-1)+}\right] = \frac{\hbar^2}{2 \cdot m_e} \cdot \left(\frac{Z \cdot q^2 \cdot m_e}{\hbar} \right)^2 - Z \cdot q^2 \cdot \frac{Z \cdot q^2 \cdot m_e}{\hbar^2} = -\frac{m_e}{2} \cdot \left(\frac{Z \cdot q^2}{\hbar} \right)^2 = -\frac{q^2 \cdot Z^2}{2 \cdot r_B} = -E_R \cdot Z^2$$

$$\boxed{E\left[A^{(Z-1)+}\right] = -E_R \cdot Z^2}$$

3.4

In the case of $A^{(Z-1)+}$ ion captures a second electron

potential energy of both electrons $= -2 \cdot \frac{Z \cdot q^2}{r_0}$

kinetic energy of the two electrons $= 2 \cdot \frac{p^2}{2 \cdot m} = \frac{\hbar^2}{m_e \cdot r_0^2}$

potential energy due to interaction between the two electrons $= \frac{q^2}{\left|\vec{r}_1 - \vec{r}_2\right|} = \frac{q^2}{2 \cdot r_0}$

$$E\left[A^{(Z-2)+}\right] = \frac{\hbar^2}{m_e \cdot r_0^2} - \frac{2 \cdot Z \cdot q^2}{r_0^2} + \frac{q^2}{2 \cdot r_0}$$

total energy is lowest when $\frac{dE}{dr_0} = 0$

hence

$$0 = -\frac{2 \cdot \hbar^2}{m_e \cdot r_0^3} + \frac{2 \cdot Z \cdot q^2}{r_0^3} - \frac{q^2}{2 \cdot r_0^2}$$

hence

$$\frac{1}{r_0} = \frac{q^2 \cdot m_e}{2 \cdot \hbar^2} \cdot \left(2 \cdot Z - \frac{1}{2}\right) = \frac{1}{r_B} \cdot \left(Z - \frac{1}{4}\right)$$

$$E\left[A^{(Z-2)+}\right] = \frac{\hbar^2}{m_e} \cdot \left(\frac{q^2 \cdot m_e}{2 \cdot \hbar^2}\right)^2 - \frac{q^2 \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{\hbar} \cdot \frac{q^2 \cdot m_e \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{2 \cdot \hbar}$$

$$E\left[A^{(Z-2)+}\right] = -\frac{m_e}{4} \cdot \left[\frac{q^2 \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{\hbar}\right]^2 = -\frac{m_e \cdot \left[q^2 \cdot \left(Z - \frac{1}{4}\right)\right]^2}{\hbar^2} = -\frac{q^2 \cdot \left(Z - \frac{1}{4}\right)^2}{\hbar^2}$$

this gives

$$\boxed{E\left[A^{(Z-2)+}\right] = -2 \cdot E_R \cdot \left(Z - \frac{1}{4}\right)^2}$$

3.5

The ion $A^{(Z-1)+}$ is at rest when it captures the second electron also at rest before capturing. From the information provided in the problem, the frequency of the photon emitted is given by

$$\nu = \frac{\omega}{2\cdot\pi} = \frac{2{,}057\cdot 10^{17}}{2\cdot\pi}\ \mathrm{Hz}$$

The energy equation can be simplified to $E\left[A^{(Z-1)+}\right] - E\left[A^{(Z-2)+}\right] = \hbar\cdot\omega = h\cdot\nu$

that is

$$-E_R\cdot Z^2 - \left[-2\cdot E_R\cdot\left(Z-\frac{1}{4}\right)^2\right] = \hbar\cdot\omega$$

putting in known numbers follows

$$2{,}180\cdot 10^{-18}\cdot\left[-Z^2 + 2\cdot\left(Z-\frac{1}{4}\right)^2\right] = 1{,}05\cdot 10^{-34}\cdot 2{,}607\cdot 10^{17}$$

this gives

$$Z^2 - Z - 12{,}7 = 0$$

with the physical sensuous result $Z = \dfrac{1+\sqrt{1+51}}{2} = 4{,}1$

$$\boxed{\text{This implies } Z = 4\text{, and that means Beryllium}}$$