Solution

1.1

1.1.1

Let v be the velocity of the ion towards the laser source relative to the laser source,
ν' the frequency of the laser light as observed by the observer moving with the ion (e.g. in the frame in which the velocity of the ion is 0) and
ν the frequency of the laser light as observed by the observer at rest with respect to the laser source.

Classical formula for Doppler´s effect is given as

$$\nu' = \nu \cdot \left(1 + \frac{v}{c}\right) \qquad (1)$$

Let ν^* be the frequency absorbed by an ion (characteristic of individual ions) and
ν_L be the frequency of the laser light used to excite an ion at rest,
hence:

$\nu^* = \nu_L$

For a moving ion, the frequency used to excite ions must be lower than ν^*.

Let ν_H be the frequency used to excite the moving ion.

When no accelerating voltage is applied

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
ν_H	0	ν^*	λ_1
ν_L	$v = 6 \cdot 10^3$ m/s	ν^*	λ_2

$\nu_L < \nu_H$

$\nu_L = \nu^*$

Calculation of frequency ν_H absorbed by moving ions.

$$\nu^* = \nu_L \cdot \left(1 + \frac{v}{c}\right) \qquad \text{where } \nu^* = \nu_H = 5 \cdot 10^{14} \text{ Hz and } v = 6 \cdot 10^3 \text{ m/s} \qquad (2)$$

The difference in the values of the frequency absorbed by the stationary ion and the ion moving with the velocity v $\qquad \Delta\nu = \nu_H - \nu_L$

The difference in the values of the wavelengths absorbed by the stationary ion and the ion moving with the velocity v $\qquad \Delta\lambda = \lambda_L - \lambda_H$

(higher frequency implies shorter wavelength)

$$\lambda_L - \lambda_H = \frac{c}{\nu_L} - \frac{c}{\nu_H}$$

from (2)

$$\lambda_L - \lambda_H = \frac{c}{\nu^*}\cdot\left(1+\frac{v}{c}\right) - \frac{c}{\nu^*} = \frac{v}{\nu^*}$$

In this case

$$\lambda_L - \lambda_H = \frac{6\cdot 10^3}{5\cdot 10^{14}}\ \text{m} = 12\cdot 10^{-3}\ \text{nm}$$

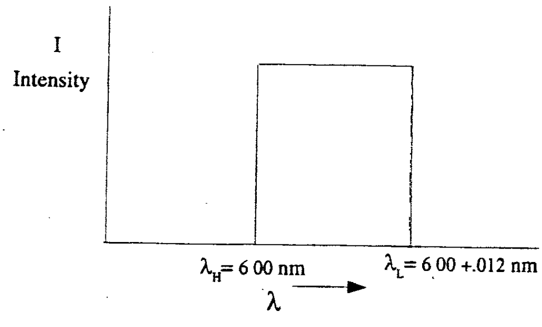

Fig. 19.3' Spectrum of laser light used to excite ions

1.1.2

The formula for calculation of ν′ as observed by the observer moving towards light source based on the principle of the theory of special relativity,

$$\nu' = \nu\cdot\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

where v is the magnitude of the velocity of the observer towards the light source,
ν′ is the frequency absorbed by the ion moving with the velocity v towards the light source (also observed by the observer moving with velocity v towards the laser source) and
ν is the frequency of laser light as observed by an observer at rest.

(To put in a metaphoric way, the moving ion "sees" the laser light of frequency ν′ even though the scientist who operates the laser source insists that he is sending a laser beam of frequency ν).

$$\nu' = \nu\cdot\sqrt{\left(1+\frac{v}{c}\right)\cdot\left(1+\frac{v}{c}+\frac{v^2}{c^2}+\ldots\ldots\ldots\ldots\right)} = \nu\cdot\sqrt{\left(1+\frac{v}{c}\right)^2+\left(1+\frac{v}{c}\right)\cdot\frac{v^2}{c^2}+\ldots\ldots\ldots\ldots}$$

$$\nu' = \nu\cdot\left(1+\frac{v}{c}\right)\cdot\left[1+\frac{v^2}{c^2}\cdot\frac{1}{1+\frac{v}{c}}+\ldots\ldots\ldots\right]^{\frac{1}{2}} = \nu\cdot\left(1+\frac{v}{c}\right)\cdot\left[1+\frac{v^2}{2\cdot c^2}\cdot\frac{1}{\left(1+\frac{v}{c}\right)}+\ldots\ldots\ldots\right]$$

The second term in the brackets represents the error if the classical formula for Doppler′s effect is employed.

$$\frac{v}{c} = 2\cdot 10^{-5}$$

$$\frac{v^2}{2\cdot c^2}\cdot\frac{1}{\left(1+\frac{v}{c}\right)} = \frac{1}{2}\cdot\frac{4\cdot 10^{-10}}{1+2\cdot 10^{-5}} \approx 2\cdot 10^{-10}$$

The error in the application of classical formula for Doppler′s effect however is of the order of the factor 2.10^{-10}. This means that classical formula for Doppler′s effect can be used to analyze the problem without loosing accuracy.

1.2 When acceleration voltage is used

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
ν_H'	v_H'	$\nu^* = 5 \,.\, 10^{14}$ Hz	λ_H'
ν_L'	v_L'	$\nu^* = 5 \,.\, 10^{14}$ Hz	λ_L'

Lowest limit of the kinetic energy of ions $\frac{1}{2}\cdot m\cdot(v_L')^2 = e\cdot U$ and $v_L' = \sqrt{\frac{2\cdot e\cdot U}{m}}$

Highest limit of the kinetic energy of ions $\frac{1}{2}\cdot m\cdot(v_H')^2 = \frac{1}{2}\cdot m\cdot v^2 + e\cdot U$

and $v_H' = \sqrt{v^2 + \frac{2\cdot e\cdot U}{m}}$

Spectrum width of velocity spectrum $\boxed{v_H' - v_L' = \sqrt{v^2 + \frac{2\cdot e\cdot U}{m}} - \sqrt{\frac{2\cdot e\cdot U}{m}}}$ .... (3)

(Note that the final velocity of accelerated ions is not the sum of v and $\sqrt{\frac{2\cdot e\cdot U}{m}}$ as velocity changes with time).

In equation (3) if $\sqrt{\frac{2\cdot e\cdot U}{m}}$ is negligibly small, the change in the width of the spectrum is negligible, by the same token of argument if $\sqrt{\frac{2\cdot e\cdot U}{m}}$ is large or approaches ∞, the width of the spectrum of the light used in exciting the ions becomes increasingly narrow and approaches 0.

1.3

Given two energy levels of the ion, corresponding to wavelength $\lambda^{(1)} = 600$ nm and $\lambda^{(2)} = 600+10^{-2}$ nm

For the sake of simplicity, the following sign notations will be adopted:

The superscript in the bracket indicates energy level (1) or (2) as the case may be. The sign $'$ above denotes the case when accelerating voltage is applied, and also the subscripts H and L apply to absorbed frequencies (and also wavelengths) correspond to the high velocity and low velocity ends of the velocity spectrum of the ion beam respectively.

The subscript following λ (or ν) can be either 1 or 2, with number 1 corresponding to lowest velocity of the ion and number 2 the highest velocity of the ion. When no accelerating voltage is applied, the subscript 1 implies that minimum velocity of the ion is 0, and the highest velocity of the ion is 6000 m/s. If accelerating voltage U is applied, number 1 indicates that the wavelength of laser light pertains to the ion of lowest velocity and number 2 indicates the ion of the highest velocity.

Finally the sign * indicates the value of the wavelength (λ^*) or frequency (ν^*) absorbed by the ion (characteristic absorbed frequency).

When no accelerating voltage is applied:

For the first energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(1)}$	0	$\nu^{(1)*} = 5 \cdot 10^{14}$ Hz	$\lambda_1^{(1)}$
$\nu_L^{(1)}$	$v=6.10^3$ m/s	$\nu^{(1)*} = 5 \cdot 10^{14}$ Hz	$\lambda_2^{(1)}$

$\nu_H^{(1)*} = \nu_L^{(1)*} = \nu^{(1)*} = 5 \cdot 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $= \nu_H^{(1)} - \nu_L^{(1)}$
Differences of wavelengths of laser light used to excite ions $= \lambda_L^{(1)} - \lambda_H^{(1)}$

$$\frac{v}{\nu_L^{(1)*}} = \frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}$$

For the second energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(2)}$	0	$\nu^{(2)*} = 5 \cdot 10^{14}$ Hz	$\lambda_H^{(2)}$
$\nu_L^{(2)}$	$v = 6000$ m/s	$\nu^{(2)*} = 5 \cdot 10^{14}$ Hz	$\lambda_L^{(2)}$

$\nu_H^{(2)*} = \nu_L^{(2)*} = \nu^{(2)*} = 5 \cdot 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $\nu_H^{(2)} - \nu_L^{(2)}$
Differences in wavelengths of laser light used to excite ions $\lambda_L^{(2)} - \lambda_H^{(2)}$

This gives $\frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}$

Hence the spectra of laser light (absorption spectrum) used to excite an ion at two energy levels overlap as shown in fig. 19.4.

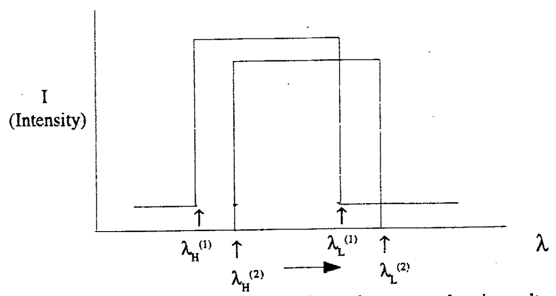

Fig. 19.4 Spectrum of laser light used to excite ions when no accelerating voltage is applied(Absorption Spectrum)

When accelerating voltage is applied:

Let $\lambda_H^{(1)\prime}$ and $\lambda_L^{(1)\prime}$ be the range of the wavelengths used to excite ions in the first energy level, when accelerating voltage is applied. (Note the prime sign to denote the situation in which the accelerating voltage is used), and let $\lambda_H^{(2)\prime}$ and $\lambda_L^{(2)\prime}$ represent the range of the wavelengths used to excite ions in the second energy level also when an accelerating voltage is applied.

Condition for the two spectra not to overlap:

$$\lambda_H^{(2)\prime} \geq \lambda_L^{(1)\prime} \quad \text{(see fig. 19.4)} \qquad (4)$$

(Keep in mind that lower energy means longer wavelengths and vice versa).

From condition (3): $$\lambda_L - \lambda_H = \frac{v}{\nu^*} \qquad (5)$$

The meanings of this equation is if the velocity of the ion is v, the wavelength which the ion "sees" is λ_L, when λ_H is the wavelength which the ion of zero-velocity "sees".

Equation (5) may be rewritten in the context of the applications of accelerating voltage in order for the two spectra of laser light will not overlap as follows:

$$\lambda_L^{(N)\prime} - \lambda_H^{(N)\prime} = \frac{v'}{\nu^*} \quad \text{where N is the order of the energy level} \qquad (6)$$

The subscript L relates λ to lowest velocity of the ion which "sees" frequency ν*. The lowest velocity in this case is $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ and the subscript H relates λ to the highest velocity of the ion, in this case $\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$.

Equation (6) will be used to calculate

- width of velocity spectrum of the ion accelerated by voltage U
- potential U which results in condition given by (4)

Let us take up the second energy level (lower energy level of the two ones) of the ion first:

$$\lambda_L^{(2)\prime} - \lambda_H^{(2)} = \frac{v'}{\nu^*} \qquad (7)$$

substitute

$$v' = \sqrt{\frac{2 \cdot e \cdot U}{m}}$$

$\lambda_H^{(1)} = 600 + 10^{-3}$ nm

$\nu^* = 5 \, . \, 10^{14}$ Hz

v = 0 m/s

$$\lambda_H^{(2)\prime} = (600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \qquad (8)$$

Considering the first energy level of the ion

$$\lambda_L^{(1)'} - \lambda_H^{(1)} = \frac{v'}{\nu^*} \quad \text{(9)}$$

In this case

$$v' = \sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$$

$\nu^* = 5 \,.\, 10^{14}$ Hz

v = 6000 m/s

$\lambda_H^{(1)} = 600 \,.\, 10^{-9}$ m

$$\lambda_L^{(1)'} = 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \quad \text{(10)}$$

Substitute $\lambda_H^{(2)'}$ from (8) and $\lambda_L^{(1)'}$ from (10) in (4) one gets

$$(600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \geq 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} - \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \sqrt{36 \cdot 10^6 + 2 \cdot 4 \cdot 10^6 \cdot U} - \sqrt{2 \cdot 4 \cdot 10^6 \cdot U}$$

assume that U is of the order of 100 and over,

then
$$\sqrt{8 \cdot 10^6 \cdot U} \cdot \left(1 + \frac{9}{4 \cdot U}\right) - \sqrt{8 \cdot 10^6 \cdot U} \leq 500$$

$$\frac{1}{\sqrt{2 \cdot U}} \cdot 9 \cdot 10^3 \leq 500$$

$$\sqrt{2 \cdot U} \geq 324$$

$$\boxed{U \geq 162 \text{ V}}$$

The minimum value of accelerating voltage to avoid overlapping of absorption spectra is approximately 162 V