Answer Q3

Equations of motion:

$$m\frac{d^2u_1}{dt^2} = k(u_2 - u_1) + k(u_3 - u_1)$$
$$m\frac{d^2u_2}{dt^2} = k(u_3 - u_2) + k(u_1 - u_2)$$
$$m\frac{d^2u_3}{dt^2} = k(u_1 - u_3) + k(u_2 - u_3)$$

Substituting $u_n(t) = u_n(0)\cos\omega t$ and $\omega_o^2 = \frac{k}{m}$:

$$(2\omega_o^2 - \omega^2)u_1(0) - \omega_o^2 u_2(0) - \omega_o^2 u_3(0) = 0 \quad \text{(a)}$$
$$-\omega_o^2 u_1(0) + (2\omega_o^2 - \omega^2)u_2(0) - \omega_o^2 u_3(0) = 0 \quad \text{(b)}$$
$$-\omega_o^2 u_1(0) - \omega_o^2 u_2(0) + (2\omega_o^2 - \omega^2)u_3(0) = 0 \quad \text{(c)}$$

Solving for $u_1(0)$ and $u_2(0)$ in terms of $u_3(0)$ using (a) and (b) and substituting into (c) gives the equation equivalent to

$$(3\omega_o^2 - \omega^2)^2 \omega^2 = 0$$
$$\omega^2 = 3\omega_o^2,\ 3\omega_o^2 \text{ and } 0$$
$$\omega = \sqrt{3}\omega_o,\ \sqrt{3}\omega_o \text{ and } 0$$

(ii) Equation of motion of the n'th particle:

$$m\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + k(u_{n-1} - u_n)$$
$$\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + \omega_o^2(u_{n-1} - u_n)$$

$n = 1,2\ldots\ldots N$

Substituting $u_n(t) = u_n(0)\sin\left(2ns\frac{\pi}{N}\right)\cos\omega_s t$

$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = \omega_o^2\left[\sin\left(2(n+1)s\frac{\pi}{N}\right) - 2\sin\left(2ns\frac{\pi}{N}\right) + \sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\frac{1}{2}\sin\left(2(n+1)s\frac{\pi}{N}\right) + \sin\left(2ns\frac{\pi}{N}\right) - \frac{1}{2}\sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right) - \sin\left(2ns\frac{\pi}{N}\right)\right]$$
$$\therefore \omega_s^2 = 2\omega_o^2\left[1 - \cos\left(2s\frac{\pi}{N}\right)\right]: \quad (s = 1,2,\ldots\ldots N)$$

As $2\sin^2\theta = 1 - \cos 2\theta$

This gives

$$\omega_s = 2\omega_o \sin\left(\frac{s\pi}{N}\right) \quad (s = 1,2,\ldots N)$$

ω_s can have values from 0 to $2\omega_o = 2\sqrt{\frac{k}{m}}$ when $N \to \infty$; corresponding to range $s = 1$ to $\frac{N}{2}$.

(iv) For s'th mode

$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2(n+1)s\frac{\pi}{N}\right)}$$

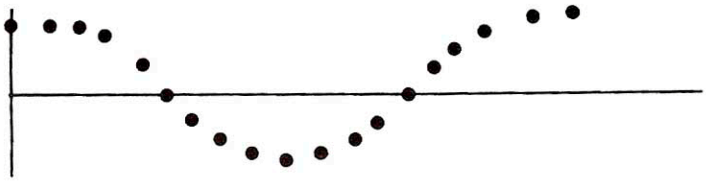
$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right) + \cos\left(2ns\frac{\pi}{N}\right)\sin\left(2s\frac{\pi}{N}\right)}$$

(a) For small ω, $\left(\frac{s}{N}\right) \approx 0$, thus $\cos\left(2ns\frac{\pi}{N}\right) \cong 1$ and $= \sin\left(2ns\frac{\pi}{N}\right) \approx 0$, and so $\frac{u_n}{u_{n+1}} \cong 1$.

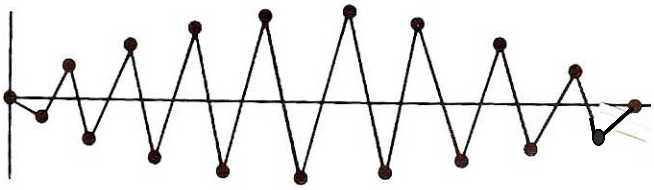
(b) The highest mode, $\omega_{\max} = 2\omega_o$, corresponds to $s = N/2$

$$\therefore \frac{u_n}{u_{n+1}} = -1 \quad \text{as} \quad \frac{\sin(2n\pi)}{\sin(2(n+1)\pi)} = -1$$

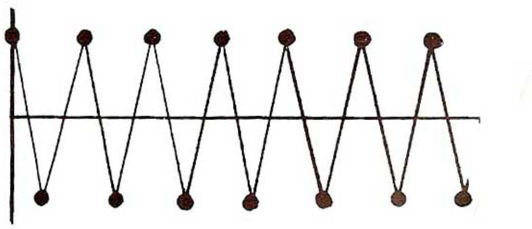
Case (a)



Case (b)
N odd

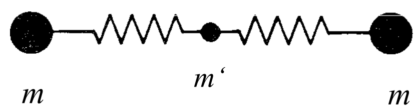


N even



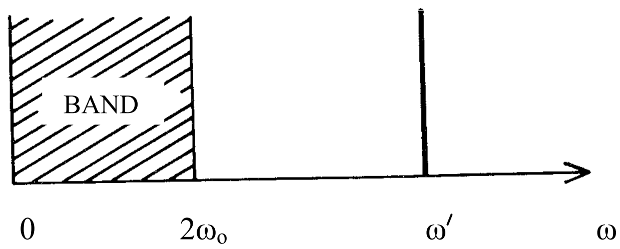
(vi) If $m' << m$, one can consider the frequency associated with m' as due to vibration of m' between two adjacent, much heavier, masses which can be considered stationary relative to m'.

The normal mode frequency of m', in this approximation, is given by



$$m'\ddot{x} = -2kx$$
$$\omega'^2 = \frac{2k}{m}$$
$$\omega' = \sqrt{\frac{2k}{m'}}$$

For small m', ω' will be much greater than ω_{max},



DIATOMIC SYSTEM

More light masses, m', will increase the number of frequencies in region of ω' giving a band-gap-band spectrum.