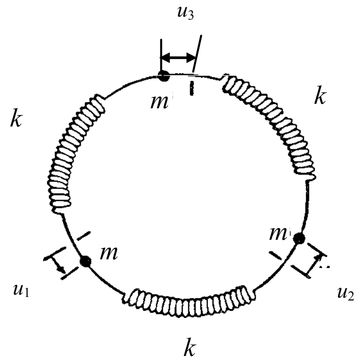
Q3

Three particles, each of mass m, are in equilibrium and joined by unstretched massless springs, each with Hooke's Law spring constant k. They are constrained to move in a circular path as indicated in Figure 3.1.

Figure 3.1



(i) If each mass is displaced from equilibrium by small displacements u_1, u_2 and u_3 respectively, write down the equation of motion for each mass.

(ii) Verify that the system has simple harmonic solutions of the form

$$u_n = a_n \cos \omega t \,,$$

with accelerations, $(-\omega^2 u_n)$ where a_n $(n = 1,2,3)$ are constant amplitudes, and ω, the angular frequency, can have 3 possible values,

$$\omega_o\sqrt{3}, \omega_o\sqrt{3} \text{ and } 0. \text{ where } \omega_o^2 = \frac{k}{m}.$$

(iii) The system of alternate springs and masses is extended to N particles, each mass m is joined by springs to its neighbouring masses. Initially the springs are unstretched and in equilibrium. Write down the equation of motion of the nth mass (n = 1,2...N) in terms of its displacement and those of the adjacent masses when the particles are displaced from equilibrium.

Show that
$$u_n(t) = a_s \sin\left(\frac{2ns\pi}{N} + \phi\right)\cos \omega_s t,$$

are oscillatory solutions where s = 1, 2,...N , n = 1, 2, ...N and where ϕ is an arbitrary phase, providing the angular frequencies are given by

$$\omega_s = 2\omega_o \sin\left(\frac{s\pi}{N}\right),$$

where a_s $(s = 1,......N)$ are constant amplitudes independent of n.

State the range of possible frequencies for a chain containing an infinite number of masses.

(iv) Determine the ratio

$$u_n / u_{n+1}$$

for large N, in the two cases:

(a) low frequency solutions

(b) $\omega = \omega_{\max}$, where $\omega_{\max}$ is the maximum frequency solution.

Sketch typical graphs indicating the displacements of the particles against particle number along the chain at time t for cases (a) and (b).

(v) If one of the masses is replaced by a mass $m' \ll m$ estimate any major change one would expect to occur to the angular frequency distribution.

Describe qualitatively the form of the frequency spectrum one would predict for a diatomic chain with alternate masses m and m' on the basis of the previous result.

Reminder

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$2\sin^2 A = 1 - \cos 2A$$