Problem 3: Hot-air-balloon

Consider a hot-air balloon with fixed volume $V_B = 1.1\ m^3$. The mass of the balloon-envelope, whose volume is to be neglected in comparison to V_B, is $m_H = 0.187$ kg.

The balloon shall be started, where the external air temperature is $\vartheta_1 = 20\ ^oC$ and the normal external air pressure is $p_o = 1.013 \cdot 10^5$ Pa. Under these conditions the density of air is $\rho_1 = 1.2\ kg/m^3$.

a) What temperature ϑ_2 must the warmed air inside the balloon have to make the balloon just float?

b) First the balloon is held fast to the ground and the internal air is heated to a steady-state temperature of $\vartheta_3 = 110\ ^oC$. The balloon is fastened with a rope.

   Calculate the force on the rope.

c) Consider the balloon being tied up at the bottom (the density of the internal air stays constant). With a steady-state temperature $\vartheta_3 = 110\ ^oC$ of the internal air the balloon rises in an isothermal atmosphere of 20 °C and a ground pressure of $p_0 = 1.013 \cdot 10^5$ Pa. Which height h can be gained by the balloon under these conditions?

d) At the height h the balloon (question c)) is pulled out of its equilibrium position by 10 m and then is released again.
   Find out by qualitative reasoning what kind of motion it is going to perform!

Solution of problem 3:

a) Floating condition:

   The total mass of the balloon, consisting of the mass of the envelope m_H and the mass of the air quantity of temperature ϑ_2 must equal the mass of the displaced air quantity with temperature $\vartheta_1 = 20\ ^oC$.

$$V_B \cdot \rho_2 + m_H = V_B \cdot \rho_1$$

$$\rho_2 = \rho_1 - \frac{m_H}{V_B} \qquad (1)$$

Then the temperature may by obtained from

$$\frac{\rho_1}{\rho_2} = \frac{T_2}{T_1},$$

$$T_2 = \frac{\rho_1}{\rho_2} \cdot T_1 = 341.53 \text{ K} = 68.38 \text{ °C} \tag{2}$$

b) The force F_B acting on the rope is the difference between the buoyant force F_A and the weight force F_G:

$$F_B = V_B \cdot \rho_1 \cdot g \ - \ (V_B \cdot \rho_3 + m_H) \cdot g \tag{3}$$

It follows with $\rho_3 \cdot T_3 = \rho_1 \cdot T_1$

$$F_B = V_B \cdot \rho_1 \cdot g \cdot \left(1 - \frac{T_1}{T_3}\right) - m_H \cdot g = 1{,}21 \text{ N} \tag{4}$$

c) The balloon rises to the height h, where the density of the external air ρ_h has the same value as the effective density ρ_{eff}, which is evaluated from the mass of the air of temperature $\vartheta_3 = 110\ ^\circ\text{C}$ (inside the balloon) and the mass of the envelope m_H:

$$\rho_{eff} = \frac{m_2}{V_B} = \frac{\rho_3 \cdot V_B + m_H}{V_B} = \rho_h = \rho_1 \cdot e^{-\frac{\rho_1 \cdot g \cdot h}{p_0}} \tag{5}$$

Resolving eq. (5) for h gives: $h = \frac{p_o}{\rho_1 \cdot g} \cdot \ln \frac{\rho_1}{\rho_{eff}} = 843 \text{m}$ (6).

d) For *small* height differences (10 m in comparison to 843 m) the exponential pressure drop (or density drop respectively) with height can be approximated by a linear function of height. Therefore the driving force is proportional to the elongation out of the equilibrium position.

This is the condition in which harmonic oscillations result, which of course are damped by the air resistance.