Solution

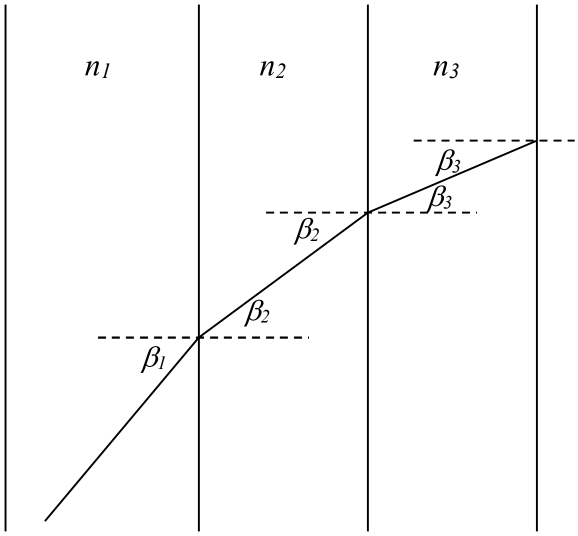

Fig. 2

Consider a light ray passing through a system of parallel plates with different refractive indexes – Fig. 2. From the Snell law we have

$$\frac{\sin\beta_2}{\sin\beta_1} = \frac{n_1}{n_2}$$

i.e.

$$n_2 \sin\beta_2 = n_1 \sin\beta_1 .$$

In the same way we get

$$n_3 \sin\beta_3 = n_2 \sin\beta_2 , \text{ etc.}$$

Thus, in general:

$$n_i \sin\beta_i = \text{const.}$$

This relation does not involve plates thickness nor their number. So, we may make use of it also in case of continuous dependence of the refractive index in one direction (in our case in the x direction).

Consider the situation shown in Fig. 3.

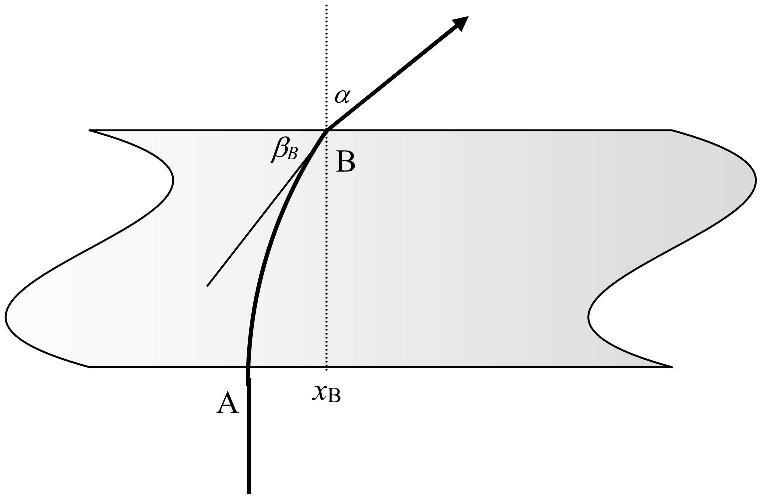

Fig. 3

At the point A the angle $\beta_A = 90°$. The refractive index at this point is n_0. Thus, we have

$$n_A \sin\beta_A = n_B \sin\beta_B ,$$
$$n_0 = n_B \sin\beta_B .$$

Additionally, from the Snell law applied to the refraction at the point B, we have

$$\frac{\sin\alpha}{\sin(90° - \beta_B)} = n_B .$$

Therefore

$$\sin\alpha = n_B \cos\beta_B = n_B\sqrt{1-\sin^2\beta_B} = \sqrt{n_B^2 - (n_B \sin\beta_B)^2} = \sqrt{n_B^2 - n_0^2}$$

and finally

$$n_B = \sqrt{n_0^2 + \sin^2\alpha} .$$

Numerically

$$n_B = \sqrt{\left(\frac{12}{10}\right)^2 + \left(\frac{5}{10}\right)^2} = 1.3$$

The value of x_B can be found from the dependence $n(x)$ given in the text of the problem. We have

$$n_B = n(x_B) = \frac{n_0}{1 - \frac{x_B}{R}},$$

$$x_B = R\left(1 - \frac{n_0}{n_B}\right),$$

Numerically

$$x_B = 1 \text{ cm}.$$

The answer to the third question requires determination of the trajectory of the light ray. According to considerations described at the beginning of the solution we may write (see Fig. 4):

$$n(x)\sin\beta(x) = n_0 .$$

Thus

$$\sin\beta(x) = \frac{n_0}{n(x)} = \frac{R - x}{R} .$$

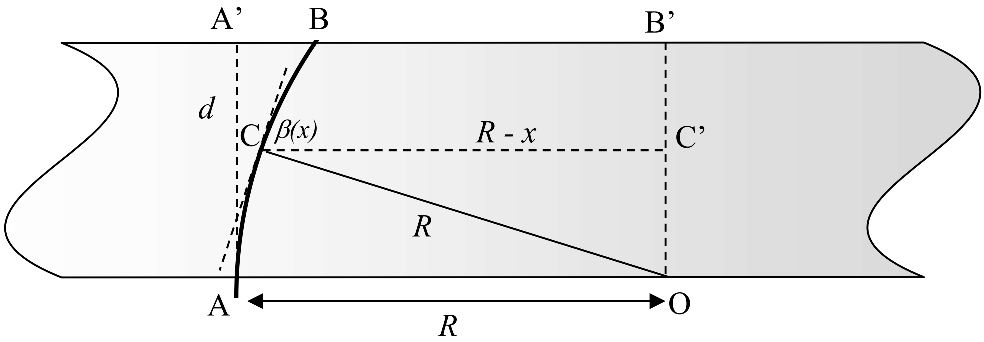

Fig. 4

Consider the direction of the ray crossing a point C on the circle with radius R and center in point O as shown in Fig. 4. We see that

$$\sin\angle \text{COC'} = \frac{R - x}{R} = \sin\beta(x) .$$

Therefore, the angle $\angle$COC' must be equal to the angle $\beta(x)$ formed at the point C by the light ray and CC'. It means that at the point C the ray must be tangent to the circle. Moreover, the ray that is tangent to the circle at some point must be tangent also at farther points. Therefore, the ray cannot leave the circle (as long as it is inside the plate)! But at the beginning the ray (at the point A) is tangent to the circle. Thus, the ray must propagate along the circle shown in Fig. 4 until reaching point B where it leaves the plate.

Already we know that A'B = 1 cm. Thus, B'B = 12 cm and from the rectangular triangle BB'O we get

$$d = \mathrm{B'O} = \sqrt{13^2 - 12^2}\ \mathrm{cm} = 5\ \mathrm{cm}.$$

The shape of the trajectory $y(x)$ can be determined also by using more sophisticated calculations. Knowing $\beta(x)$ we find $\mathrm{tg}\,\beta(x)$:

$$\mathrm{tg}\,\beta(x) = \frac{R-x}{\sqrt{R^2-(R-x)^2}}.$$

But $\mathrm{tg}\,\beta(x)$ is the derivative of $y(x)$. So, we have

$$\frac{dy}{dx} = \frac{R-x}{\sqrt{R^2-(r-x)^2}} = \frac{d}{dx}\left(\sqrt{R^2-(R-x)^2}\right).$$

Thus

$$y = \sqrt{R^2-(R-x)^2} + const$$

Value of *const* can be found from the condition

$$y(0) = 0.$$

Finally:

$$y = \sqrt{R^2-(R-x)^2}.$$

It means that the ray moves in the plate along to the circle as found previously.

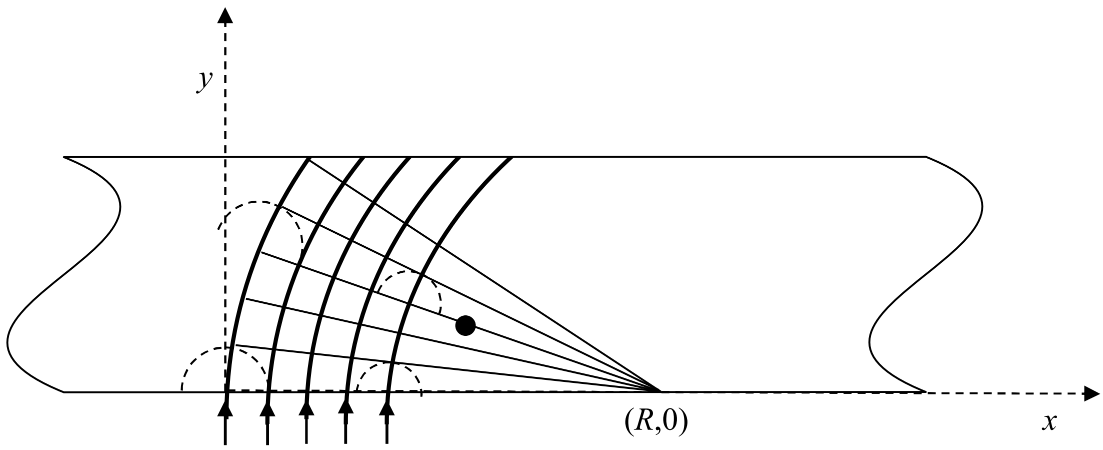

Fig. 5

Now we will present yet another, already the third, method of proving that the light in the plate must move along the circle.

We draw a number of straight lines (inside the plate) close to each other and passing trough the point (R,0) - Fig. 5. From the formula given in the text of the problem it follows that the refraction index on each of these lines is inversely proportional to the distance to the point (R,0). Now we draw several arcs with the center at (R,0). It is obvious that the geometric length of each arc between two lines is proportional to the distance to the point (R,0).

It follows from the above that the optical path (a product of geometric length and refractive index) along each arc between the two lines (close to each other) is the same for all the arcs.

Assume that at +-certain moment t the wave front reached one of the lines, e.g. the line marked with a black dot in Fig. 5. According to the Huygens principle, the secondary sources on this line emit secondary waves. Their envelope forms the wave front of the real wave at some time $t + \Delta t$. The wave fronts of secondary waves, shown in Fig. 5, have different geometric radii, but - in view of our previous considerations - their optical radii are exactly the same. It means that at the time $t + \Delta t$ the new wave front will correspond to one of the lines passing trough (R,0). At the beginning the wave front of the light coincided with the x axis, it means that inside the plate the light will move along the circle with center at the point (R,0).

The solutions were marked according to the following scheme (draft):

1.	Proof of the relation $n \sin \beta =$ const	up to 2 points
2.	Correct description of refraction at points A and B	up to 2 points
3.	Calculation of x_B	up to 1 point
4.	Calculation of d	up to 5 points