Solution Problem 2

a) We consider argon an ideal mono-atomic gas and the collisions of the atoms with the piston perfect elastic. In such a collision with a fix wall the speed $\vec{v}$ of the particle changes only the direction so that the speed $\vec{v}$ and the speed $\vec{v}'$ after collision there are in the same plane with the normal and the incident and reflection angle are equal.

$$v_n^{'} = -v_n ,\ v_t^{'} = v_t \qquad (1)$$

In the problem the wall moves with the speed $\vec{u}$ perpendicular on the wall. The relative speed of the particle with respect the wall is $\vec{v} - \vec{u}$. Choosing the Oz axis perpendicular on the wall in the sense of $\vec{u}$, the conditions of the elastic collision give:

$$(\vec{v} - \vec{u})_z = -(\vec{v}' - \vec{u})_z \ ,\ (\vec{v} - \vec{u})_{x,y} = (\vec{v}' - \vec{u})_{x,y} ;$$

$$v_z - u = -(v_z^{'} - u) \ ,\ v_z^{'} = 2u - v_z ,\ v_{x,y}^{'} = v_{x,y} \qquad (2)$$

The increase of the kinetic energy of the particle with mass m_o after collision is:

$$\frac{1}{2}m_o v^{'2} - \frac{1}{2}m_o v^2 = \frac{1}{2}m_o\left(v_z^{'2} - v_z^2\right) = 2m_o u(u - v_z) \cong -2m_o u v_z \qquad (3)$$

because u is much smaller than v_z.

If n_k is the number of molecules from unit volume with the speed component v_{zk}, then the number of molecules with this component which collide in the time dt the area dS of the piston is:

$$\frac{1}{2}n_k v_{zk} dt dS \qquad (4)$$

These molecules will have a change of the kinetic energy:

$$\frac{1}{2}n_k v_{zk} dt dS(-2m_o u v_{zk}) = -m_o n_k v_{zk}^2 dV \qquad (5)$$

where $dV = u dt dS$ is the increase of the volume of gas.

The change of the kinetic energy of the gas corresponding to the increase of volume dV is:

$$dE_c = -m_o dV \sum_k n_k v_{zk}^2 = -\frac{1}{3} n m_o \bar{v}^2 dV \qquad (6)$$

and:

$$dU = -\frac{2}{3} N \frac{m_o \bar{v}^2}{2} \cdot \frac{dV}{V} = -\frac{2}{3} U \frac{dV}{V} \qquad (7)$$

Integrating equation (7) results:

$$UV^{2/3} = const. \qquad (8)$$

The internal energy of the ideal mono-atomic gas is proportional with the absolute temperature T and the equation (8) can be written:

$$TV^{2/3} = const. \qquad (9)$$

b) The oxygen is in contact with a thermostat and will suffer an isothermal process. The internal energy will be modified only by the adiabatic process suffered by argon gas:

$\Delta U = \nu C_V \Delta T = m c_V \Delta T$ (10)

where ν is the number of kilomoles. For argon $C_V = \frac{3}{2}R$.

For the entire system L=0 and $\Delta U = Q$.

We will use indices 1, respectively 2, for the measures corresponding to argon from cylinder A, respectively oxygen from the cylinder B:

$$\Delta U = \frac{m_1}{\mu_1}\cdot\frac{3}{2}\cdot R\left(T_1^{'} - T_1\right) = Q = \frac{m_1}{\mu_1}\cdot\frac{3}{2}RT_1\left[\left(\frac{V_1}{V_1^{'}}\right)^{2/3} - 1\right] \quad (11)$$

From equation (11) results:

$$T_1 = \frac{2}{3}\cdot\frac{\mu_1}{m_1}\cdot\frac{Q}{R}\cdot\frac{1}{\left(\frac{V_1}{V_1^{'}}\right)^{2/3} - 1} = 1000K \quad (12)$$

$$T_1^{'} = \frac{T_1}{4} = 250K \quad (13)$$

For the isothermal process suffered by oxygen:

$$\frac{\rho_2^{'}}{\rho_2} = \frac{p_2^{'}}{p_2} \quad (14)$$

$p_2^{'} = 2{,}00atm = 2{,}026\cdot 10^5 N/m^2$

From the equilibrium condition:

$$p_1^{'} = p_2^{'} = 2atm \quad (15)$$

For argon:

$$p_1 = p_1^{'}\cdot\frac{V_1^{'}}{V_1}\cdot\frac{T_1}{T_1^{'}} = 64atm = 64{,}9\cdot 10^5 N/m^2 \quad (16)$$

$$V_1 = \frac{m_1}{\mu_1}\cdot\frac{RT_1}{p_1} = 1{,}02m^3, V_1^{'} = 8V_1 = 8{,}16m^3 \quad (17)$$

c) When the valve is opened the gases intermix and at thermal equilibrium the final pressure will be $p^{'}$ and the temperature T. The total number of kilomoles is constant:

$$\nu_1 + \nu_2 = \nu^{'}, \frac{p_1^{'}V_1^{'}}{RT_1^{'}} + \frac{p_2^{'}V_2^{'}}{RT} = \frac{p\left(V_1^{'} + V_2^{'}\right)}{RT} \quad (18)$$

$$p_1^{'} + p_2^{'} = 2atm, T_2 = T_2^{'} = T = 300K$$

The total volume of the system is constant:

$$V_1 + V_2 = V_1^{'} + V_2^{'}, \quad \frac{V_2^{'}}{V_2} = \frac{\rho_2}{\rho_2^{'}}, \quad V_2^{'} = \frac{V_2}{2} = 7{,}14m^3 \quad (19)$$

From equation (18) results the final pressure:

$$p = p_1^{'}\cdot\frac{1}{V_1 + V_2}\cdot\left(V_1^{'}\cdot\frac{T}{T_1^{'}} + V_2^{'}\right) = 2{,}2atm = 2{,}23\cdot 10^5 N/m^2 \quad (20)$$